

# Urban transport in polycentric cities\*

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## Abstract

This paper considers commuting patterns in polycentric cities. We model a city with one CBD and two SBDs. Roads are assumed to connect every business districts while the radial public transit does not connect the two SBDs directly. First, we show how an optimal pricing scheme can restore efficiency. Second, we compare three different administration regimes: public (roads and rail are administrated by a public operator), semi public (roads are administrated by a private company) and private. Third, we consider the possibility to open a new transit line between the two SBDs and show that it is not always Pareto improving (the Downs-Thompson paradox). It depends on the relative impact on crowding within the transit line and on the network externality for public transit.

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# 1 Introduction

An abundant literature documents the transition from monocentric to polycentric city structures. Along with that transition, the proportion of standard commuting, from Suburban Business Districts to the Central Business District (CBD and SBDs hereafter) has declined while reverse commuting (from CBD to SBD) and suburb-to-suburb (SBD to SBD) commuting have increased (see Aguilera et al. (2009) for a discussion of these facts). Giuliano & Small (1993) already discussed job-housing (in-)balance and excess commuting, defined as the difference between actual commute and the commute required to access job when people are efficiently located.

In this paper, we study the behaviour of commuters living in a given business district and having to commute to another business district. The issues and emergence of spatial mismatch and multicentricity are not directly addressed as we focus on commuting behaviours between business districts. The considered metropolitan structure is reduced to a single CBD and two SBDs. Direct roads connect every business districts but, transit lines connect only the CBD to both SBDs. There is no transit line between the SBDs. Therefore, commuting between the two SBDs implies either to drive or to pass through the CBD with public transit. In this context, we address three main questions. First, we look at the optimal commuting scheme with the current transportation network and show that applying appropriate pricing scheme allows to restore efficiency. To improve the analysis, we consider various administration regimes. A semi-private regime, where transit is administrated by a public operator and roads by a private one, and a fully private regime where both operators (of transit and roads) are private. Second, we consider the possibility to invest in a new transit line connecting the two SBDs. Third and finally, we discuss the possibility of improving transit frequencies and synchronization between transit lines.

In the next Section, we present the model. The equilibrium and the optimality conditions are derived in Section 3. We consider two distinct administrative regimes for roads and transit in Section 3.3 and address further questions in Section 4: the possibility to decide on the frequencies and to invest in a new transit line connecting the two SBDs.

## 2 The model

We consider a city with a single central place (called CBD) and two suburban areas (SBD). There are radial and circumferential roads directly connecting the suburban areas and the central place, but there are only radial transit lines connecting suburban areas to the city center. So, to make a trip from a suburban area to the other suburban area, the transit user must travel first to the city center and then to her destination.

This is a simplified figure of realistic situations with a central and main place and multiple suburban areas. A number of cities has developed a radial transit network where transit lines extend from the city center to suburban districts. The case of the Paris region is illustrative here.

The goal of the paper is to evaluate transport costs in this context and explore policy reforms that can increase urban welfare (pricing, extension of transit network and changes in transit frequencies). These issues are of great importance for many agglomerations where transit develop first through radial lines. As a consequence most trips made between suburban areas are made by car (see Aguilera (2005), for instance). These are relatively long distance trips, potentially at the origin of important CO<sub>2</sub> emissions and congestion. The objective of many reforms in the transport sector is the reduction of the usage of cars in metropolitan areas through a better supply of public transport services. Some regions like Paris have ongoing projects for the development of circumferential transit lines.

For simplicity, we reduce the analysis to one CBD and two SBDs. The main district is denoted  $C$ , for “Center”, the first SBD is denoted  $S$  for “south” and the second  $E$  for “East”. Fig. 1 illustrates the geometry of this city. Dashed lines are used for public transit while continuous lines represent roads. As suggested by the arrows, we only look at commuting behaviour from the South to the Center and the East and from the Center to the East and not the other ways around. As long as we assume that the flows of commuters are symmetric, this does not affect our results.

The modal choice consists, for commuters, to decide to use either a car on the road (private mode, denoted  $R$ ) or transit (public mode, denoted  $T$ ). There are three groups of commuters, distinguished by their residential location (origin) and working place (desti-

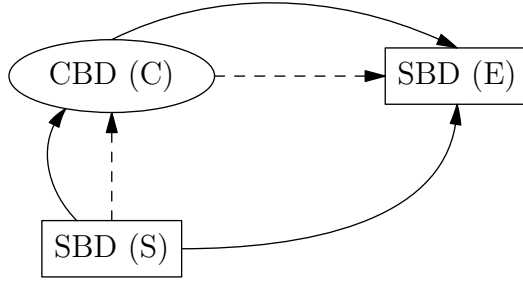


Figure 1: The city structure (to be completed).

nation). The group sizes are respectively denoted  $N_{sc}$ ,  $N_{ce}$  and  $N_{se}$ , where the subscripts  $(ij)$  refer to the origin-destination pairs. Thus,  $i = C, S$  and  $j = S, E$ . We define  $n_{ij}^M$  as the number of users from  $i$  to  $j$ , using mode  $M$ , with  $M = R, T$ . We have  $n_{ij}^R + n_{ij}^T = N_{ij}$ . As shown in Fig. 1 there is a total of five links, two transit links and three road links. On each link  $(ij)$ , the total number of users of mode  $M$ , is denoted  $u_{ij}^M$ . More precisely, we have  $u_{ij}^R = n_{ij}^R$  for  $(ij) \in \{(sc), (ce), (se)\}$  and  $u_{ij}^T = n_{ij}^T + n_{se}^T$  for  $(ij) \in \{(sc), (ce)\}$ .

Notice that these trips are at odd with the prediction of the monocentric city, which suggests that each worker commutes to the nearest business place. The word “wasteful commuters” was introduced by Hamilton (1982) to characterize these trips. Indeed, Hamilton found that wasteful commuting are very large in metropolitan areas he observed (Los Angeles). The empirical conclusion was first criticized by White (1988), before is was confirmed by Small & Song (1992). It is now considered as a major critics to the monocentric city model (cf. Brueckner 2011). We do not address this question and do not consider why a household may locate “inefficiently”, but take the empirical evidence that the so called wasteful commuting exists.

Transport is costly for all modes. Road users directly drive to their destination. It is assumed that those who drive between suburbs do not use radial roads. We discuss this restriction below. Congestion on roads depends on (and is increasing in) the number of users on that roads. Crowding in transit depends on the number of users of transit, where the cost function depends on the number of users and the service frequency. Transport cost functions are defined for each link connecting pairs of business places  $i$  and  $j$ . They are denoted  $c_{ij}^M$  and depend on the number of users of the same link, denoted  $u_{ij}^M$ . For a given link  $(i, j)$ , the generalized transport cost using the road is given by

$$C_{ij}^R(u_{ij}^R) = \tau_{ij} + c_{ij}^R(u_{ij}^R) \quad \text{for} \quad (ij) = \{(sc), (ce) \text{ or } (se)\} \quad (1)$$

where  $\tau_{ij}$  denotes a road toll imposed to the users and  $c_{ij}^R(u_{ij}^R) = F_{ij}^R + \tilde{c}_{ij}^R(u_{ij}^R)$  is the monetary value of the time spent for the commute. It is the sum of a free-flow travel cost,  $F_{ij}^R$ , and the user cost due to congestion,  $\tilde{c}_{ij}^R(u_{ij}^R)$ . The free-flow travel cost encompasses the monetary value of the travel time and the vehicle operating cost. We assume that  $\tilde{c}^R(u^R)$  is twice differentiable with  $\partial \tilde{c}^R(u^R)/\partial u^R > 0$  and make no specific assumption on the second-order derivative at this stage. For public transport, the generalized transport cost on a single link  $(ij)$ <sup>1</sup> for the users is given by

$$C_{ij}^T(u_{ij}^T, f_{ij}) = p_{ij} + c_{ij}^T(u_{ij}^T, f_{ij}) \quad \text{for} \quad (ij) = \{(sc) \text{ or } (ce)\} \quad (2a)$$

where  $p_{ij}$  is the transport fares paid by the users of public mode and  $c_{ij}^T(u_{ij}^T, f_{ij}) = \frac{c_w}{2f_{ij}} + F_{ij}^T + \tilde{c}_{ij}^T(u_{ij}^T, f_{ij})$  is the monetary value of the time spent in commuting by train. It is composed of the waiting time at the train station,  $\frac{c_w}{2f_{ij}}$ , the crowding-free travel cost,  $F_{ij}^T$ , and the monetary value of crowding when there are  $u_{ij}^T$  passengers in the train and a train frequency of  $f_{ij}$  on the line. Assuming that transit passengers arrive uniformly at the station (which suggests that they do not use timetables), it is given by  $c_w/2f_{ij}$  where  $c_w$  is the monetary value of the maximal waiting time between two trains. We assume that crowding costs increases with passengers and decreases with frequency, i.e.

$$\frac{\partial \tilde{c}^T(u^T, f)}{\partial f} < 0 \quad \text{and} \quad \frac{\partial \tilde{c}^T(u^T, f)}{\partial u^T} > 0,$$

and make no specific assumptions on the second-order partial derivatives. The last term corresponds to the waiting time at the station, which depends on frequencies.

On the  $(se)$  link, there is no direct transit line. Commuters must transit through  $C$ . Their travel cost is given by

$$C_{se}^T(u_{sc}^T, u_{ce}^T, f_{sc}, f_{ce}) = \beta(p_{sc} + p_{ce}) + c_{sc}^T(u_{sc}^T, f_{sc}) + c_{ce}^T(u_{ce}^T, f_{ce}) - \Gamma(\alpha). \quad (2b)$$

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<sup>1</sup>A single link is defined as either  $(sc)$  or  $(ce)$ . When using transit, the link  $(se)$  is composed of two single links.

Transit users of the  $\{se\}$  line are assumed to pay a fare proportional to  $(p_{sc} + p_{ce})$ . In the real world, fares are generally higher than  $\max\{p_{sc}, p_{ce}\}$  and smaller or equal to  $(p_{sc} + p_{ce})$ . We impose no restriction on the parameter  $\beta$  which captures these situations, but could also captures other cases. For instance,  $\beta < 0$  would correspond to a subsidy provided to these users. The last term reflects the effect on the waiting time of a synchronization between the arrivals of trains ( $sc$ ) and the departures of trains ( $ce$ ) at station  $C$ . Although transit users bear the monetary value of the time spent on each line, we add a term that captures the possibility to improve the connectivity between the two lines. We assume that the higher the  $\alpha$ , the better the coordination between the two lines. If the same train goes from  $S$  to  $E$  with a stop in  $C$ ,  $\alpha = 1$  and the transit user does not incur the waiting time in  $C$ . If  $\alpha = 0$ , there is no coordination between the two lines and the transit user incurs the full cost of waiting at the two train station. An additional switching cost ( $c_w$ ) may be considered. The value of  $\alpha$  varies between 0 and 1 and corresponds to the quality of synchronization. It is possible that the same trains starts at  $S$ , goes to  $C$  and continues to  $E$ . In this case there is no switching cost ( $c_s = 0$ ) and a perfect synchronization between the two trains ( $\alpha = 0$ ). A small value of these parameters may reflect the intermediate stop at station  $C$ . So, parameters  $c_s$  and  $\alpha$  can be chosen to describe a variety of situations. Timetables for the services are assumed to adopt uniform schedules, and train loadings are equal on all vehicles so it is straightforward to compute both waiting time and crowding cost. All costs are expressed for a whole trip.

Transport sector is administrated by one or two operators (one for each mode) who can be either private (profit maximizing) or public (welfare maximizing). The choice variables for the commuters is the transportation mode. The operator of the roads, can decide to impose a toll on a given link, and the operator of public transport decides the fares and service frequencies. For public transport, there is also the possibility to extend the transit network to make a direct connection between the suburban areas  $S$  and  $E$ . We first consider that there is no cost of administrating the roads of the rails. In this context, frequencies ( $f_{ij}$ ) and coordination of the transit system ( $\alpha$ ) are considered exogeneous. We relax these assumptions in an extension and use a cost of providing operating vehicles and to synchronize the two lines.

### 3 Equilibrium, optimum and administration regimes

In this section, we use a very general cost function to characterize the equilibrium and optimal solutions. We compare the two and provide the conditions for the optimum to be decentralized. Then we consider various administration regimes. In the public regime, both roads and transit are assumed to be managed by the social planner. Under a semi-public administration, roads are administrated by a private operator and public transit by the social planner. In the duopoly scenario, both public transit and roads are administrated by private operators. We compare pricing scheme and welfare under the three scenarii.

In this Section, service frequencies ( $f_{ij}$ ) and the synchronization ( $\alpha$ ) are assumed to be exogeneous and free of charge.<sup>2</sup> As a consequence, we focus on the social optimum for commuters. In the next Section, we relax these assumptions by looking at transport supply.

#### 3.1 Equilibrium

Workers are assumed to commute from their living to their working places using either private cars or public transit. As demand is perfectly inelastic, we can rewrite  $n_{ij}^T$  as functions of  $n_{ij}^R$ . We have

$$n_{sc}^R = N_{sc} - n_{sc}^T, \quad n_{ce}^R = N_{ce} - n_{ce}^T, \quad n_{se}^R = N_{se} - n_{se}^T,$$

and stay with three endogenous variables:  $n_{sc}^T, n_{ce}^T, n_{se}^T$ .

This problem meets the Wardrop equilibrium conditions. Therefore, for a given set of commuters, the user cost in the private mode is equal to the user cost in the public mode. If a mode were not used it must be associated to higher cost. In an interior solution, when both the public and the private modes are used, we have, in equilibrium:  $C_{ij}^T(u_{ij}^T) = C_{ij}^R(u_{ij}^R) \forall (ij) = (sc), (ce)$  and  $(se)$ , i.e.

$$\tau_{sc} + c_{sc}^R(u_{sc}^R) = p_{sc} + c_{sc}^T(u_{sc}^T) \tag{3a}$$

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<sup>2</sup>As  $f_{ij}$  is assumed to be exogenous in this section, for the ease of reading, it will be removed from the expressions. For instance,  $C_{ij}^T(u_{ij}^T, f_{ij})$  will be denoted  $C_{ij}^T(u_{ij}^T)$ . This assumption will be relaxed in the next section and  $f_{ij}$  will be reintroduced in the equations.



$$\tau_{ce} + c_{ce}^R(u_{ce}^R) = p_{ce} + c_{ce}^T(u_{ce}^T) \quad (3b)$$

$$\tau_{se} + c_{se}^R(u_{se}^R) = \beta(p_{sc} + p_{ce}) + c_{sc}^T(u_{sc}^T) + c_{ce}^T(u_{ce}^T) - \Gamma(\alpha) \quad (3c)$$

Note that in a corner solution, some modes (or links) may not be used and the above condition associated to these modes (or links) do not hold. We have the following result.

**Proposition 1 (Equilibrium)** *The problem of modal choice has at least an equilibrium: (i) If the set of equations  $C_{ij}^T(u_{ij}^T) = C_{ij}^R(u_{ij}^R)$ , for  $(ij) = (sc), (ce)$  and  $(se)$ , has a feasible solution then it is the unique interior equilibrium (where each group uses both transport modes). (ii) In all other cases, the problem has at least a corner solution and some groups do not use both transport mode.*

Proofs are in Appendix A. To obtain this result, we transform the equilibrium problem into a minimization problem for commuters, with constraints for the variables. We show that the objective function is convex. The stability of the equilibrium is straightforward in this problem since an additional user will always increase the user cost. So, any deviation is costly, and, at the equilibrium, every user supports her minimum travel cost.

## 3.2 Optimum

The total cost is the sum of the users' costs and the operators' cost. An optimum is reached when the total cost is minimized. As there is no operating costs for the roads, the total cost is the sum of users' costs and the cost of operating the transit system. This latter depends on the frequency of the services and the effort to coordinate the two lines. As these values are exogenously fixed for the moment, the operator's cost is fixed. Since transit fares and road tolls are redistributed to the population they are welfare-neutral. Therefore, the objective is to minimize the following social-cost function for  $n_{sc}^T$ ,  $n_{ce}^T$  and  $n_{se}^T$ :

$$\sum_{ij=sc,ce,se} u_{ij}^R c_{ij}^R(u_{ij}^R) + \sum_{ij=sc,ce} u_{ij}^T c_{ij}^T(u_{ij}^T) - n_{se}^T \Gamma(\alpha). \quad (4)$$

The endogenous variables should satisfy the usual constraints, i.e.

$$0 \leq n_{sc}^T \leq N_{sc}, 0 \leq n_{ce}^T \leq N_{ce} \text{ and } 0 \leq n_{se}^T \leq N_{se}.$$

We have the following result.

**Proposition 2 (Optimum)** *There exist traffic flows  $n_{ij}^M$  that satisfy the constraints and minimize the total cost function in (4). If there is an interior solution and if  $\partial^2 c_{ij}^M / \partial u^2 \geq 0$ , the solution is unique.*

Note that various standard formulations of congestion (like the BPR or the quadratic formulations, for instance) satisfy the second-order condition for  $c_{ij}^T(u)$ . If one takes into account a cost function that considers the availability of seats a non-convex structure may appear. The so called MAS formulation (cf. de Palma et al. 2015), for example, has a second order derivative that is negative on some intervals. In the general case, we can have multiple solutions. We show in the proof that the above condition on the second-order derivative is a sufficient condition for a unique solution. It guarantees the convexity of the objective function.

The equilibrium is generally distinct from the optimal solution. The first-order conditions with respect to the objective function in (4) yields

$$c_{ij}^T(u_{ij}^T) + u_{ij}^T \frac{\partial c_{ij}^T(u_{ij}^T)}{\partial n_{ij}^T} = c_{ij}^R(u_{ij}^R) - u_{ij}^R \frac{\partial c_{ij}^R(u_{ij}^R)}{\partial n_{ij}^T} \quad (5a)$$

for groups  $ij = \{sc\}, \{ce\}$ , and

$$\sum_{ij=sc,ce} \left( c_{ij}^T(u_{ij}^T) + u_{ij}^T \frac{\partial c_{ij}^T(u_{ij}^T)}{\partial n_{ij}^T} \right) - \Gamma(\alpha) = c_{se}^R(u_{se}^R) - u_{se}^R \frac{\partial c_{se}^R(u_{se}^R)}{\partial n_{se}^T} \quad (5b)$$

for  $ij = \{se\}$ , which is a usual statement that the optimum distribution of users is such that the social marginal costs, not the private marginal costs, are equal for all the used alternatives.

### 3.3 Administration regimes

In most cases, urban transit systems are under the control of public authorities because of their cost structure, and because this sector is not in general able to yield positive benefits. To what extent the public operator should set tolls or fares to induce a decrease of the transportation cost could be questioned as it should also consider the externalities produced by the transport system.

In this section, we consider various scenarios regarding the management of the transport system. We start considering a fully public administration where the social planner

administrates both the roads and the transit system. We the turn to the case were the administration of roads is delegated to a private operator. Finally, we look at a fully private administration where each transport system is administrated by a private operator. We refer to this case as the duopoly administration.

### 3.3.1 Public administration

By comparing (5) with the equilibrium conditions (3), we can see that the optimum can be decentralized if

$$p_{ij}^{op} - \tau_{ij}^{op} = u_{ij}^T \frac{\partial c_{ij}^T}{\partial n_{ij}^T} + u_{ij}^R \frac{\partial c_{ij}^R}{\partial n_{ij}^T} \text{ for } (ij) = (sc) \text{ and } (ce) \quad (6a)$$

$$\beta^{op}(p_{se}^{op} + p_{ce}^{op}) - \tau_{se}^{op} = \sum_{ij=sc,ce} u_{ij}^T \frac{\partial c_{ij}^T}{\partial n_{se}^T} + u_{se}^R \frac{\partial c_{se}^R}{\partial n_{se}^T}, \quad (6b)$$

where  $(op)$  is used for the optimum. We have the following result.

**Corollary 1 (Decentralisation of the optimum)** *Any pricing scheme such that the differences between the traffic fares and the road tolls correspond to the differences between the marginal social cost of crowding and of congestion ensures to reach the optimum.*

In addition, it worth to note that having the control of one of the two tools (tolls or fares) is sufficient to reach the optimum as long as the social planner can set tolls or fare such that (6a) and (6b) are satisfied. A typical regime for the decentralization of the optimum is where public transportation is unpriced and roads are tolled according to (6) (with  $p_{ij} = 0$ ). Notice that pricing public transportation only can also lead to the optimum but, it is a little more difficult to implement in practice because we have to distinguish users ( $SE$ ) from the other two groups. Generally, flat pricing of public transportation with similar fares for all groups, which is used in several cities (and debated for Paris region), will not yield the optimum without road pricing.

The external cost considered could also reflect emissions of pollutants. This would lead to higher distortions in equilibrium.

### 3.3.2 Semi-public administration

In this section, roads are assumed to be operated by a private operator whose objective function is to maximize profit. The transit system is operated by a public agent whose objective function is to minimize total transport cost.

Generally speaking, a private operator will increase his revenues by imposing higher tolls on road users. For the social planner, two scenarii reserve attention. In the first scenario, the public operator is assumed to set fares at zero. If tolls are higher than the optimal ones, then, by comparison to the first-best situation, the private mode will be under-used. In the second scenario we let the public operator increase the fares in public transport to go back to the optimum.

Operating the roads is assumed to be costless. The private operator's benefit is the sum of toll revenues collected on the three roads. It is given by

$$\pi^R(\tau_{sc}, \tau_{ce}, \tau_{se}, p_{sc}, p_{ce}) = \sum_{ij=sc,ce,se} \tau_{ij} n_{ij}^R. \quad (7)$$

He is constrained by the equilibrium choice of users described in equations (3). The first-order conditions with respect to the tolls yields  $\lambda_{ij} = -(N_{sc} - n_{sc}^T) \leq 0$  for  $\{ij\} = \{sc\}, \{ce\}, \{se\}$ , where  $\lambda_{ij}$  is the multiplier of constraint (3). Substituting in the first-order conditions with respect to  $n_{ij}^T$  shows that the tolls imposed by the private operator satisfy

$$\tau_{ij}^{sp}(\cdot) = (u_{ij}^R + u_{se}^R) \frac{\partial c_{ij}^T}{\partial n_{ij}^T} - u_{ij}^R \frac{\partial c_{ij}^R}{\partial n_{ij}^T} \geq 0 \quad (8a)$$

for  $(ij) = (sc), (ce)$  and

$$\tau_{se}^{sp}(\cdot) = \sum_{ij=sc,ce} (u_{ij}^R + u_{se}^R) \frac{\partial c_{ij}^T}{\partial n_{se}^T} - u_{se}^R \frac{\partial c_{se}^R}{\partial n_{se}^T} \geq 0. \quad (8b)$$

Notice that  $\partial c_{ij}^R / \partial n_{ij}^T < 0$ . So, comparing (8) with (6), we see that for the same level of transit fares, the road operator imposes tolls that are higher than the optimum level. The following proposition states that when public transport is unpriced, the private operator imposes tolls that higher than the optimum tolls.

**Proposition 3 (Unpriced public transport)** *If the social planner keeps the transit free ( $p_{ij} = 0$ ), road tolls that are imposed by the private operator are higher than those that decentralize the optimum, i.e.  $\tau_{ij}^{sp}(p_{ij} = 0) \geq \tau_{ij}^{op}(p_{ij} = 0)$  for  $(ij) = (sc), (ce), (se)$ . In this case, public transport is over-used by all user groups, i.e.  $n_{sc}^{T,sp} \geq n_{sc}^{T,O}$ ,  $n_{ce}^{T,sp} \geq n_{ce}^{T,sp}$  and  $n_{se}^{T,sp} \geq n_{se}^{T,O}$ .*

The case where roads are tolled and public transportation are subsidized is a prevalent idea for economic efficiency. Proposition 3 confirms the fact that road tolls will reduce the usage of roads but lead to an excess usage of public transportation. A similar conclusion can be found in Kraus (2012) and Kilani et al. (2014). With unpriced roads it is optimal to reduce fares below the marginal social cost. But, when pigouvian tolls are imposed on road users it is optimal to raise fares so that crowding costs are endogenized. The next proposition shows that it is always possible for the public operator to reach the optimum.

**Proposition 4 (Optimum and pricing scheme in the semi-public regime)** *The public operator can reach the optimum by setting fares*

$$p_{ij}^{sp}(\cdot) = (N_{ij} + N_{se}) \frac{\partial c_{ij}^T}{\partial n_{ij}^T} \geq 0 \text{ for } (ij) = (sc), (ce) \text{ and} \quad (9a)$$

$$\beta = \frac{\sum_{ij=sc,ce} (N_{ij} + N_{se}) \frac{\partial c_{ij}^T}{\partial n_{se}^T}}{p_{sc} + p_{ce}} \geq 0. \quad (9b)$$

*The private operator, then, imposes roads tolls that are higher than those he would impose if public transit was free.*

The semi-public regime induces a strategic competition between the public and the private operator. The strategic variables in this duopoly, which are fares (choice variable for the public operator) and road tolls (choice variable for the private operator), are strategic complements. As a result the equilibrium is characterized by excess pricing, but the public operator is able to reach the optimum as shown in Proposition 4. In practice, this may raise the issue of acceptability since users of the modes pay higher fares and higher tolls. The reaction function of the public operator is obtained from Eq. (9), and the reaction function for the private operator is obtained from Eq. (8).

From the expressions of the fares in Eq. (4) it is clear that user price is proportional to the marginal social cost. In the case where there is no crowding in public transportation the road operator would impose optimal tolls. The next proposition states this result.

**Corollary 2 (No crowding)** *If there is no crowding in public transport ( $\partial c_{ij}^T / \partial n_{ij}^T = 0$ ) the private operator imposes social optimum road tolls when public transportation are kept unpriced.*

In this case, the optimum can be easily achieved through the privatization of roads and by setting no fares in public transit. An illustration in a simplified network and a brief discussion of that proposition is provided in appendix B. This result dates back to Knight (1924) and is usually quoted for the advocacy of privatization of road management. The scope of this result, however, is not general. It is well known that it is sensitive to two main assumptions (cf. Lindsey 2012), both adopted in our framework. The first one is the elastic demand in the network, and the second one is the homogeneity of the users. Indeed, even if users are distinguished with respect to their origin-destination pairs they have the same time values and they perceive the same magnitude of discomfort. If one of these two assumptions is not satisfied, the private operator will impose a non optimal road tolls.

### 3.3.3 Private administration

In this section, we consider two distinct and competing private operators free to set road prices and transit fares. The first one is operating the roads, as in the preceding section, and the second one is operating public transportation. The operators do not incur any cost and their profit is equal to their revenue. So, the road operator maximizes equation (7), as in the semi-public regime, and the public transport operator chooses transit fares to maximize

$$\pi^T(\tau_{sc}, \tau_{ce}, \tau_{se}, p_{sc}, p_{ce}, \beta) = p_{sc}n_{sc}^T + p_{ce}n_{ce}^T + \beta(p_{ce} + p_{ce})n_{se}^T \quad (10)$$

In both cases, and since we assume an interior solution, equilibrium conditions in (3) constraint each operator. The first-order conditions for the road operator are given by (8) and the first order conditions for the transit operator are (where the index  $du$  stands for “duopoly”):

$$\tau_{ij}^{du} + c_{ij}^R + n_{ij}^T \frac{\partial c_{ij}^R}{\partial n_{ij}^T} = c_{ij}^T + u_{ij}^T \frac{\partial c_{ij}^T}{\partial n_{ij}^T} \text{ for } (ij) = (sc), (ce) \text{ and} \quad (11a)$$

$$\tau_{se}^{du} + c_{se}^R + n_{se}^T \frac{\partial c_{se}^R}{\partial n_{se}^T} = c_{sc}^T + c_{ce}^T + u_{sc}^T \frac{\partial c_{sc}^T}{\partial n_{se}^T} + u_{ce}^T \frac{\partial c_{ce}^T}{\partial n_{se}^T} - \Gamma(\alpha) \quad (11b)$$

Combining these first-order conditions with the equilibrium conditions, we have an expression for the fares:

$$p_{ij}^{du} = u_{ij}^T \frac{\partial c_{ij}^T}{\partial n_{ij}^T} - n_{ij}^T \frac{\partial c_{ij}^R}{\partial n_{ij}^T} \quad (12a)$$

$$\beta = \frac{u_{sc}^T \frac{\partial c_{sc}^T}{\partial n_{se}^T} + u_{ce}^T \frac{\partial c_{ce}^T}{\partial n_{se}^T} - n_{se}^T \frac{\partial c_{se}^R}{\partial n_{se}^T}}{p_{sc} + p_{ce}}. \quad (12b)$$

Optimal tolls set by the private operator on roads are given by equations (8). These reaction functions between two monopoly competing to attract a inelastic demand leads to a strategic complementarity between transit fares and road tolls.

## 4 Public transport supply

In the previous section, we looked at the pricing scheme of urban transport under various administration regimes. We will now turn to the analysis of transport supply by addressing two key questions. First, we study public transport supply at given network structure. More precisely, we look at the provided service frequencies and the coordination between the two trains. Second, discuss the consequences of changing the network by building a new transit line between the two subcenters.

For many cities whose public transport network is radial, that question is of primary interest. Following the example of Paris, some cities have projects which consists of changing the metro network with the construction of new peripheral transit lines.

### 4.1 Endogenous service frequency and synchronization

Consider that the operator chooses the frequency of the services and the level of the synchronization between the two trains ( $\alpha$ ) at the central station. The cost of providing operating vehicles and their synchronization is given by

$$\kappa \sum_{ij=sc,ce} f_{ij} + v(\alpha), \quad (13)$$

where  $\kappa > 0$  denotes the unit cost of operating a vehicle (the summation is done over all links), and  $v(\alpha)$  (with  $v'(\cdot) > 0$ ) is the cost of deploying an effort to synchronize the two trains. The above term has to be added to the social cost function (4).

The first order conditions (5) remain unchanged but additional conditions for frequencies and coordination emerge. At optimum, we must have:

$$\kappa = -u_{ij}^T \frac{\partial c_{ij}^T(u_{ij}^T, f_{ij})}{\partial f_{ij}} \text{ for } ij = sc \text{ and } ce, \text{ and} \quad (14a)$$

$$v'(\alpha) = n_{se}^T \Gamma'(\alpha), \quad (14b)$$

whose interpretation is straightforward. The marginal cost of increasing frequencies or improving synchronization (left hand sides) must equal their social marginal benefits (right hand sides). Note that these expressions implicitly display positive links between service frequencies or the level of coordination and the number of transit users ( $f_{ij}^{*'}(u_{ij}^T) > 0$  and  $\alpha^{*'}(u_{ij}^T) > 0$ ). If a public operator is in charge of public transportation, optimal service frequencies and coordination are easily achieved.

If a private operator is in charge of public transport, its objective function described in (10) becomes

$$\pi^T(\tau_{sc}, \tau_{ce}, \tau_{se}, p_{sc}, p_{ce}, \beta, f_{sc}, f_{ce}, \alpha) = p_{sc} n_{sc}^T + p_{ce} n_{ce}^T + \beta(p_{ce} + p_{ce}) n_{se}^T - \kappa \sum_{ij=sc,ce} f_{ij} - v(\alpha),$$

subject to the same equilibrium conditions (3).

**Lemma 1 (Decentralization of service frequencies and coordination)** *A private operator chooses optimally service frequencies ( $f_{ij}$ ) and the level of coordination ( $\alpha$ ) between the two transit lines.*

The solutions of the Lagrangian associated to the maximization problem of the private operator leads to the same first order conditions (14) as those associated to the optimum.

## 4.2 Investment in a new $SE$ transit line

The purpose of this section is to study the consequences of building a new, direct, transit line between  $S$  and  $E$ . For simplicity, and without loss of generality, we will assume that building such a line is free of charge. Although that assumption is not realistic at all, we



will show that such a line can, in some cases, decrease social welfare. These cases would be more likely observed if we considered positive building costs. In addition, we will consider interior solutions and assume that every commuter uses the direct connection between two business centers. In other words, we assume that any commuter going from  $i$  to  $j$  uses either the direct transit line or the direct road.

In this new setup, looking at the modal choice on each link is a question extensively discussed by David & Foucart (2014). The novelty of our approach consists of looking at the welfare consequences of a modification of the transportation network. From a theoretical point of view, the main difference with the previous sections is the equilibrium condition (3c), which becomes (we neglect road prices and transit fares as our focus is on social welfare):

$$c_{ij}^R(u_{ij}^R) = c_{ij}^T(u_{ij}^T, f_{ij}), \forall \{ij\} = \{sc, ce, se\}$$

with  $u_{ij}^M = n_{ij}^M \forall \{ij\} = \{sc, ce, se\}$  and  $M = \{T, R\}$ .

The total (social) cost function can be reorganized by mode and origin-destination pairs:

$$C = n_{sc}^T c_{sc}^T + n_{ce}^T c_{ce}^T + n_{se}^T c_{se}^T + n_{sc}^R c_{sc}^R + n_{ce}^R c_{ce}^R + n_{se}^R c_{se}^R - \kappa \sum_{ij=sc,ce} f_{ij}$$

where the third term replaces  $n_{se}^T (c_{sc}^T + c_{ce}^T) - \Gamma(\alpha)$  and  $v(\alpha)$  drops as there is no more coordination issue for  $SE$  commuters using the transit lines.

The comparison between the two networks is difficult to perform formally without defining explicit functions for the cost parameters. Instead, we discuss the welfare effects on commuters of a new transit line under three scenarii. First, we assume congestion and crowding with fixed frequencies. Second, we assume congestion, no crowding in public transport and endogenous frequencies. Third and finally, we discuss the most general case, considering together congestion, crowding and endogenous frequencies. The main results for the six group of commuters are summarized in table 1.

The first scenario is easy to understand and allows to easily identify the forces at stake. With fixed frequencies, opening the new transit line reduces the number of commuters on the two other transit lines (no more  $SE$  commuters use the  $SC$  or  $CE$  transit line anymore). By assumption, these commuters are better off (by revealed preferences).

	Congestion, no crowding, fixed frequencies	Congestion, no crowding, endogenous frequencies	Congestion, crowding, endogenous frequencies
$c_{sc}^R$	↓	↑	?
$n_{sc}^R$	↓	↑	?
$c_{sc}^T$	↓	↑ ( $f_{sc} \downarrow$ )	? ( $f_{sc} \downarrow$ & crowding ↓)
$n_{sc}^T$	↑	↓	?
$c_{ce}^R$	↓	↑	?
$n_{ce}^R$	↓	↑	?
$c_{ce}^T$	↓	↑ ( $f_{ce} \downarrow$ )	? ( $f_{ce} \downarrow$ & crowding ↓)
$n_{ce}^T$	↑	↓	?
$c_{se}^R$	↓	↓	↓
$n_{se}^R$	↓	↓	↓
$c_{se}^T$	↓	↓	↓
$n_{se}^T$	↑	↑	↑
Conclusions	Pareto improving	Not Pareto improving	Pareto improving if crowding effects on costs
on transport costs	Transport costs decrease	Transport costs increase	overcome frequencies effects
	for all commuters	for $SC$ and $CE$ commuters	Impact on transport costs depend on relative impact on
		and decrease for $SE$ only.	crowding, congestion and frequencies

Table 1: Impact of opening a new  $SE$  transit line for commuters

Commuters from  $S$  to  $C$  and  $C$  to  $E$  that were using the transit are also better off as there is less crowding. Therefore, some road users will change their modal choice, reducing congestion on roads up to the point where the cost of using roads and the cost of using transit is equal again on each  $OD$ -pairs. Under this scenario, opening a new transit line is Pareto improving. Every commuter is better off.

In the second scenario, we assume that there is congestion on roads, but no crowding in transit. By assumption,  $SE$  commuters are better off with the new line. As a consequence, there are fewer transit user on the two other lines. As there is no crowding, this does not increase their welfare. On the contrary, as there are fewer commuters in transit, the frequencies decrease, increasing the cost of using transit for both,  $SC$  and  $CE$  commuters. Some commuters will change their modal choice and use the road. Congestion increase on roads and frequencies decrease again. We are back on equilibrium when the cost of both modes are equal. As compared to the original network, there are more road user and less users of transit on the  $SC$  and  $CE$  links. They all face higher costs. Only the  $SE$  commuters are better off. Under this scenario, the opening of a new transit line is very likely to decrease social welfare.

In the last considered scenario, we assume congestion, crowding and endogenous frequencies. As in the two previous scenarii, the impact on  $SE$  commuters is assumed to be positive. The impact on the  $SC$  and  $CE$  commuters depends on the relative forces of decreasing crowding (by having less commuters in transit) and decreasing frequencies. If the former dominates on both transit lines, it would be Pareto improving. If the later dominates, it will not be Pareto improving and could even be welfare decreasing (if the negative impact on  $SC$  and  $CE$  commuters is more important than positive effect on  $SE$  commuters).

To conclude, as long as frequencies are endogenous (which is a reasonable assumption) opening a new transit line between subcenters can be desirable only if crowding in public transport is an issue. Otherwise, it is likely to be welfare decreasing by reducing frequencies and increasing congestion on roads.

## 5 Linear formulation and numerical illustration

### 5.1 A linear formulation

In this section, we use a specific (linear) cost function to derive analytical solutions and study various administration regimes for roads and public transit. Given the notation above the specific formulation concerns functions  $\tilde{c}_{ij}^m$ .

For road, the cost on link  $(ij)$  is assumed to be

$$\tilde{c}_{ij}^R(u_{ij}^R) = a_{ij}^R u_{ij}^R \quad (15a)$$

where  $a_{ij}^R$  denotes the marginal external congestion cost on roads. For public transport on link  $(ij)$ , the cost function is assumed to be

$$\tilde{c}_{ij}^T(u_{ij}^T, f_{ij}) = \frac{a_{ij}^T u_{ij}^T}{f_{ij}}. \quad (15b)$$

In an interior solution, the number of users of public transport on each line ( $u_{sc}^T$  and  $u_{ce}^T$ , with  $u_{sc}^T = n_{sc}^T + n_{se}^T$  and  $u_{ce}^T = n_{ce}^T + n_{se}^T$ ) is obtained from the equilibrium conditions (3). From these solutions, it is possible to derive reaction functions between the three groups of transit users in equilibrium. We have:

$$n_{sc}^T = \rho_{sc}(n_{se}^T), \quad n_{ce}^T = \rho_{ce}(n_{se}^T) \quad \text{and} \quad n_{se}^T = \rho_{se}(n_{ce}^T, n_{sc}^T). \quad (16)$$

These reaction functions are linear, decreasing in their arguments and reflect the fact that users  $n_{sc}^T$  and  $n_{ce}^T$  compete with users  $n_{se}^T$  for transit lines.

Under the linear formulation, both the equilibrium and the optimum conditions can be written in matrix form. This will be useful to discuss the decentralization of the equilibrium. Let matrix  $A$  be given by

$$A = \begin{pmatrix} a_{sc}^R + \frac{a_{sc}^T}{f_{sc}} & 0 & \frac{a_{sc}^T}{f_{sc}} \\ 0 & a_{ce}^R + \frac{a_{ce}^T}{f_{ce}} & \frac{a_{ce}^T}{f_{ce}} \\ \frac{a_{sc}^T}{f_{sc}} & \frac{a_{ce}^T}{f_{ce}} & a_{se}^R + \frac{a_{sc}^T}{f_{sc}} + \frac{a_{ce}^T}{f_{ce}} \end{pmatrix},$$

vector  $x' = (n_{sc}^T, n_{ce}^T, n_{se}^T)$ , and vectors  $b^F$ ,  $b^A$  and  $b^\tau$  be given by

$$b^\tau = \begin{pmatrix} \tau_{sc} - p_{sc} \\ \tau_{ce} - p_{ce} \\ \tau_{se} - \beta(p_{sc} + p_{ce}) \end{pmatrix}, \quad b^A = \begin{pmatrix} a_{sc}^R N_{sc} \\ a_{ce}^R N_{ce} \\ a_{se}^R N_{se} \end{pmatrix}$$

and

$$b^F = \begin{pmatrix} F_{sc}^R - F_{sc}^T - \frac{c_w}{2f_{sc}} \\ F_{ce}^R - F_{ce}^T - \frac{c_w}{2f_{ce}} \\ F_{se}^R - F_{sc}^T - F_{ce}^T - \frac{c_w}{2f_{sc}} - \frac{c_w}{2f_{ce}} - \Gamma(\alpha) \end{pmatrix}.$$

One can show that an interior solution for the equilibrium problem can be set as a system of three equations of the form

$$Ax = b^F + b^\tau + b^A. \quad (17)$$

On the other hand, in the linear formulation described here, the first-order conditions for the minimization problem presented in (5) can be put in the matrix form  $2Ax = b^F + 2b^A$ . It follows that the optimum tolls are given by  $b^\tau = -(\frac{1}{2})b^F$ .

The problem with the linear cost, no toll and no transit fare is illustrated on Fig. 2. The three planes correspond to the reaction functions shown above. At their intersection we find the equilibrium point  $P^e$ . The other point, denoted  $P^O$  correspond to the optimum. In this example, public transit is underused and the optimum requires imposing positive tolls on the three roads. The decentralization of the optimum can be explained on the basis of this illustration. The three planes, denoted  $\mathcal{P}_{ij}$ , correspond to the three reaction functions shown above. From equation (17), we see that road tolls enter additively. As a consequence, variations in the tolls lead to parallel movements of the planes  $\mathcal{P}_{ij}$ . It is then clear that by appropriately moving the three planes the equilibrium point,  $P^e$ , can be located anywhere, and in particular at the same location as that of  $P^O$ . We will discuss a numerical example with details in section 5.2.

With the linear formulation some static comparatives can be done by directly differentiating the solutions. Let us consider users  $\{sc\}$ . An increase in  $N_{sc}$  increases  $n_{se}^T$ ,

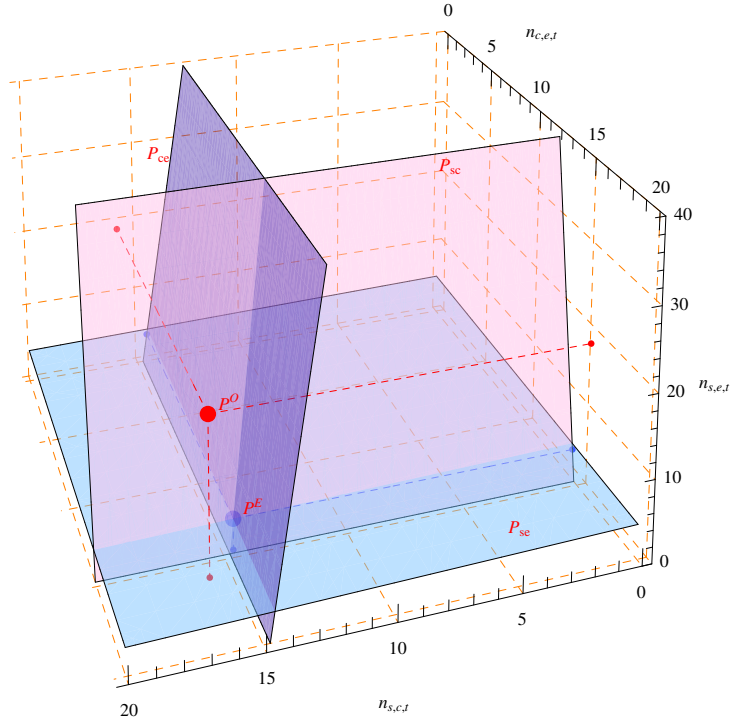


Figure 2: An illustration of the equilibrium.

while an increase in  $N_{se}$  leads to a decrease of  $n_{sc}^T$ . A less direct effect is that  $N_{ce}$  has the same impact as  $N_{sc}$ , but with smaller magnitude. Indeed, an increase in users  $\{ce\}$  will increase the usage of public transit by this group ( $n_{ce}^T$  increases) and discourages users  $\{se\}$  from using the same mode (leading to a decrease in  $n_{se}^T$ ). Overall, it has a positive impact on  $n_{sc}^T$ . An increase in the switching cost  $c_s$  reduces  $n_{se}^T$  and, by so, increases  $n_{sc}^T$  and  $n_{ce}^T$ . This is the same effect as that of parameters  $\alpha$  and  $\beta$ . An increase in the waiting cost reduces the attractiveness of the public mode. The reduction in the number of users leads to higher comfort that can attract some additional users. This rebound effect is small in this case. For  $\alpha = 1$ , an increase in  $c_w$  reduces the attractiveness of public transit for the three groups of users. For a smaller value of  $\alpha$ , an increase in  $c_w$  has always a negative impact on  $n_{se}^T$  and  $n_{ce}^T$  but may lead to an increase in  $n_{sc}^T$ .

Let us consider some particular situations. Writing down the conditions for an interior solution are complex when there is crowding in transit and congestion roads. The case of no crowding in transit is useful because it leads to simple expressions as stated in this result.

**Lemma 2 (Linear cost and no crowding in public transport)** *Assume no crowd-*

ing in public transport, i.e.  $a_{ij}^T = 0$  for  $\{ij\} = \{sc\}, \{ce\}, \{se\}$ . We have two cases:

1. if  $F_{ij}^R + a_{ij}^R N_{ij} > \frac{c_w}{2f_{ij}} + F_{ij}^T - \tau_{ij} + p_{ij} > F_{ij}^R$  for  $\{ij\} = \{sc\}$  or  $\{ce\}$ , then group  $\{ij\}$  uses both transport modes (similar condition holds for group  $\{se\}$ : if  $F_{se}^R + a_{se}^R N_{se} > \frac{c_w}{2f_{sc}} + \frac{\alpha c_w}{2f_{ce}} + F_{sc}^T + F_{ce}^T - \tau_{se} + p_{sc} + \beta p_{ce} > F_{se}^R$ , then group  $\{se\}$  uses both transport modes);
2. if this condition is not satisfied, then group  $\{ij\}$  uses only one mode. If the left part of the inequality is not satisfied, the group uses the private mode, and conversely if the right hand part of the inequality does not hold.

If the difference in the free-flow travel costs in the two modes are very important then all users will choose the same mode. Roads can be the selected mode when congestion costs produced by all users do not compensate for the high cost in public transport. This can occur when public transport has very slow speed or very low frequency. It may also occur when the correspondence is complex (for users  $se$  in this model). Since, crowding is not considered here, public transport can be the only selected mode when roads are so long or have poor qualities inducing a large generalized costs by comparison to public transport. This situation can occur only in some particular real situations. Indeed, for car owners, high congestion can motivate them to use public transport. For example, in downtown Paris more than 90% of the trips are made by public transport. The roads are very congested, but this concerns more the users travelling between Paris and the outer parts of the city. For those who travel inside Paris, they have to choose between very congested roads and frequent and fast metro services. Despite the important crowding in public transit at peak hours most users prefer public transport.

## 5.2 A numerical illustration

The following numerical illustration concerns the case of Paris. The CBD is Chatelet Les Halles. The southern SBD is Arcueil-Cachan, which is located at 9km at the South of Chatelet Les Halles. The Eastern SBD is Noisy-Champs, located at 23 km east from Chatelet Les Halles. For the moment, the best way to travel from Arcueil-Cachan to Noisy-Champs is to take the *RER B* from Arcueil-Cachan to Chatelet Les Halles and

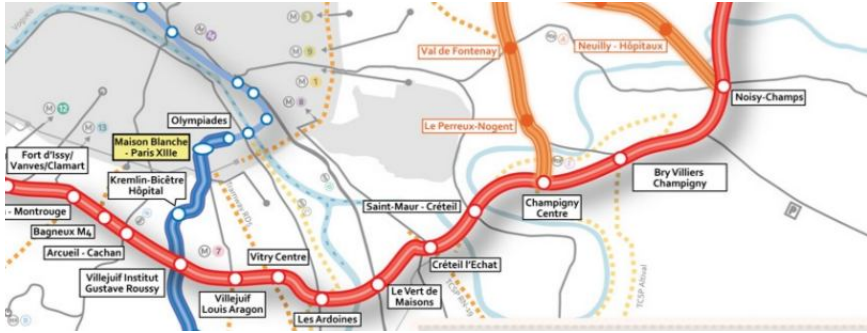


Figure 3: Part of the Grand Paris express project.

switch at Chatelet les Halles to go to Noisy-Champs with the *RER A*. Figure 3 show the relevant part of the network. We use real distances, free flow travel time as provided by Google Map and fares from the RATP. For this example, we have considered fares for single trips and standard pricing but taking account transport subscription and special fares is straightforward.

It worth to say that by 2022, in the context of the infrastructure investment, a direct line will be build between these two SBD.

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Preliminary variables: the city				
	FF-speed		Distance	
	Train	Road	Train	Road
$S \rightarrow C$	20	30	5	5
$C \rightarrow E$	20	30	5	5
$S \rightarrow E$	–	50	5	5
$\widehat{SCE}$	$\pi/2$			
Population				
$N_{sc}$	20			
$N_{ce}$	20			
$N_{se}$	20			
Roads				
$F_{sc}^R$	5		$a_{sc}^R$	1
$F_{se}^R$	5		$a_{se}^R$	1
$F_{se}^R$	8		$a_{se}^R$	1
Trains				
$F_{sc}^T$	7		$a_{sc}^T$	0
$F_{se}^T$	7		$a_{se}^T$	0
Other cost components				
$c_w$	1		$\alpha$	0.5
$c_s$	1		$\beta$	0

Table 2: Values for the numerical example. Parameters  $F_{ij}^T$  are computed from the speeds and distances.

	Transit users			Fares		Tolls			Other		Cost
	$n_{sc}^T$	$n_{ce}^T$	$n_{se}^T$	$p_{sc}$	$p_{ce}$	$\tau_{sc}$	$\tau_{ce}$	$\tau_{se}$	$\alpha$	$\beta$	
Equilibrium	14.54	14.54	3.74	0	0	0	0	0	0.5	0	8.76
Optimum	15.84	15.84	19.01	0	0	0.67	0.67	5.13	0.5	0	7.76
Second-best											
<i>Fares 1</i>	23	15.8	11.8	-2.8	-0.5	0	0	0	0.5	0	8.16
<i>Fares 2</i>	20.	20.	14.8	-1.9	-1.9	0	0	0	0.5	1	7.95
Semi-public	15.0	15.8	19.8	1.49	0.97	1.9	1.65	6.87	0.5	0	7.77
Duopoly	10.4	11.5	15.7	4.5	3.8	3.4	8.4	6.87	0.5	0	8.0

Table 3: Results for the numerical example.

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## A Proofs

### A.1 Proof of proposition 1

(WRITE THE FINAL VERSION) We state the equilibrium problem as constrained minimization one and then characterize its solutions. Following Smith (1979), the equilibrium problem can be cast a minimization of the following objective function<sup>3</sup>

$$\sum_{ij=sc,ce,se} \int_0^{u_{ij}^R} c_{ij}^R(z) dz + \sum_{ij=sc,ce} \int_0^{u_{ij}^T} c_{ij}^T(z, f_{ij}) dz + \int_0^{n_{se}^T} \left( c_s + (\alpha - 1) \frac{c_w}{f_{ce}} \right) dz, \quad (18)$$

under the constraints  $0 \leq n_{sc}^T \leq N_{sc}$ ,  $0 \leq n_{ce}^T \leq N_{ce}$  and  $0 \leq n_{se}^T \leq N_{se}$ . The existence of a solution is obtained by observing that the objective function is continuous and is it defined on a compact set (we apply the Weierstrass theorem). The first-order conditions for an interior solution is a set of three equalities. The advantage of using the minimization problem is that it handles corner solutions as well. A twice differentiation of the objective function given by Eq.(18) yields the hessian matrix

$$H = \begin{pmatrix} \frac{\partial c_{sc}^R}{\partial n_{sc}^T} + \frac{\partial c_{sc}^T}{\partial n_{sc}^T} & 0 & \frac{\partial c_{sc}^T}{\partial n_{sc}^T} \\ 0 & \frac{\partial c_{ce}^R}{\partial n_{ce}^T} + \frac{\partial c_{ce}^T}{\partial n_{ce}^T} & \frac{\partial c_{ce}^T}{\partial n_{ce}^T} \\ \frac{\partial c_{sc}^T}{\partial n_{sc}^T} & \frac{\partial c_{ce}^T}{\partial n_{ce}^T} & \frac{\partial c_{ce}^R}{\partial n_{ce}^T} + \frac{\partial c_{sc}^T}{\partial n_{sc}^T} + \frac{\partial c_{ce}^T}{\partial n_{ce}^T} \end{pmatrix}.$$

---

<sup>3</sup>Detailed expression of this problem are given in the Appendix A.1.

It is easy to see that the first two minors of  $H$  are positive as long as the first-order derivatives of the cost functions are positive. The computation of the determinant (the third minor) is little tedious. Check that it is also positive. So, the matrix  $H$  is definite positive, and it follows that the minimization problem is convex. To state the stability consider an initial equilibrium. For a given user group all used routes have the same generalized costs. If one user decides to change unilaterally his decision the generalized cost on the new road can only be higher.

## A.2 Proof of proposition 2

(WRITE THE FINAL VERSION) As in the first proof, the existence of a solution is obtained by observing that the objective function is continuous and is defined on a compact set. For the case of a unique interior equilibrium, we have to prove that the objective function in Eq. (4) is convex. The  $3 \times 3$ -hessian matrix is too large and we provided its components one by one. The first two elements in the diagonal of this matrix have

$$2 \frac{\partial c_{ij}^R}{\partial n_{ij}^R} + n_{ij}^R \frac{\partial^2 c_{ij}^R}{\partial (n_{ij}^R)^2} + 2(C_{ij}^T)_{[1,0]} + (n_{ij}^T + n_{se}^T)(C_{ij}^T)_{[2,0]} \quad \text{for } ij = sc, ce,$$

and where  $[n, m]$  subscript denote the partial derivatives of orders  $n$  and  $m$ , respectively, for the first and the second derivatives. The third and last element in the diagonal is

$$2 \frac{\partial c_{ce}^R}{\partial n_{se}^R} + n_{se}^R \frac{\partial c_{ij}^R}{\partial (n_{se}^R)^2} + 2(c_{sc}^T)_{[1,0]} + 2(c_{ce}^T)_{[1,0]} + (n_{sc}^T + n_{se}^T)(C_{sc}^T)_{[2,0]} + (n_{ce}^T + n_{se}^T)(C_{ce}^T)_{[2,0]}.$$

Elements (2, 1) and (1, 2) are zero. The other four elements are equal to

$$2(c_{ij}^T)_{[1,0]} + (n_{ij}^T + n_{se}^T)(c_{ij}^T)_{[2,0]} \quad \text{for } ij = sc, ce,$$

where  $ij = sc$  for elements (1, 3) and (3, 1), and  $ij = ce$  for elements (2, 3) and (3, 2).

When the second-order partial derivatives are positive, the condition in Proposition 2, it is clear that the first principal minor is positive. The second principal minor is a determinant of a diagonal matrix and is also positive. The computation of the third minor, the determinant of matrix  $H$ , involves a tedious computations which also shows that it is positive. We put the details of this computation in the accompanying MATHEMATICA notebook.

This proves that the total cost, the objective function, is convex. Together with the bound constraints this yields a convex program with a unique solution where the objective function reaches its minimum. Notice that the condition on the second-order partial derivatives are sufficient but not necessary. Indeed, even with negative second-order derivatives the hessian matrix remain positive definite as long as the first-order derivatives dominate.

### A.3 Proof of corollary 1

(CHECK AND WRITE THE FINAL VERSION) Consider an interior solution solution  $(n_{sc}^*, n_{ce}^*, n_{se}^*)$ . Let us ignore the second argument, the frequencies, in function  $\tilde{c}_{ij}^T$  and write them as  $\tilde{c}_{ij}^T(n_{sc}^T + n_{se}^T)$ . The set of the three equalities can be written as

$$\begin{aligned}\tau_{sc} &= \underbrace{\tilde{c}_{sc}^T(n_{sc}^T + n_{se}^T) - c_{sc}^R(n_{sc}^R)}_{h_{sc}(n_{sc}, n_{se})} \\ \tau_{ce} &= \underbrace{\tilde{c}_{ce}^T(n_{ce}^T + n_{se}^T) - c_{ce}^R(n_{ce}^R)}_{h_{ce}(n_{ce}, n_{se})} \\ \tau_{se} &= \underbrace{\tilde{c}_{ce}^T(n_{ce}^T + n_{se}^T) + \tilde{c}_{se}^T(n_{sc}^T + n_{se}^T) - c_{se}^R(n_{sc}^R)}_{h_{se}(n_{sc}, n_{ce}, n_{se})}.\end{aligned}$$

Functions  $h_{ij}^T$  compute the toll levels on each road as a function of the number of users in transit and on roads. Notice that by the construction of the cost functions, cf. Eqs. (1)-(2), the right hand members above are monotone increasing in  $n_{ij}^T$ , so functions  $h_{ij}$  are uniquely defined.

### A.4 Proof of Lemma 2

(OK) Use a simple computation to simplify equilibrium conditions and rearrange terms.

### A.5 Proof of Proposition 3

The profit of the private operator is

$$\tau_{sc}n_{sc}^R + \tau_{ce}n_{ce}^R + \tau_{se}n_{se}^R.$$

Substituting for the tolls from (3), and using  $n_{ij}^R = N_{ij} - n_{ij}^T$ , in the profit of the private operator, the latter writes as  $[c_{sc}^T - c_{sc}^R] (N_{sc} - n_{sc}^T) + [c_{ce}^T - c_{ce}^R] (N_{ce} - n_{ce}^T) + [c_{sc}^T + c_{ce}^T - c_{se}^R] (N_{se} - n_{se}^T)$ , when public transport fares are zero. This expression of the profit can be arranged to get

$$\begin{aligned} & N_{sc}c_{sc}^T - (n_{sc}^Tc_{sc}^T + n_{sc}^Rc_{sc}^R) + N_{ce}c_{ce}^T - (n_{ce}^Tc_{ce}^T + n_{ce}^Rc_{ce}^R) + \\ & N_{se}(c_{sc}^T + c_{ce}^T) - (n_{se}^T(c_{sc}^T + c_{ce}^T) + n_{se}^Rc_{se}^R) \\ & = (N_{sc} + N_{se})c_{sc}^T + (N_{se} + N_{ce})c_{se}^T - TC, \end{aligned} \quad (19)$$

where  $TC$  is the total social cost given in (4). Now consider the first-order condition with respect to  $n_{sc}^T$  in (19). It yields

$$(N_{sc} + N_{se}) \frac{\partial c_{sc}^T}{\partial n_{sc}^T} = \frac{\partial TC}{\partial n_{sc}^T}. \quad (20)$$

The left-hand side member in (20) is clearly positive, so is the right-hand member. It is straightforward to check that the first-order conditions with respect to  $n_{ce}^T$  and  $n_{se}^T$  yield similar conclusion. So the private operator sets the tolls so that the derivative of the social cost is positive. Since the cost function is convex, this is possible only when public transport is overused. The tolls set by the private operator are then higher than the optimum tolls.

## A.6 Proof of Proposition 4

Consider the profit function of the private operator. We do the same substitutions for the toll but with non zero fares. We obtain a first-order condition

$$(N_{sc} + N_{se}) \frac{\partial c_{sc}^T}{\partial n_{sc}^T} - p_{sc} = \frac{\partial TC}{\partial n_{sc}^T},$$

which is slightly different from (20). A similar condition is obtained for  $n_{ce}^T$  and  $n_{se}^T$ . So, when the public operator sets fares given by (9), the first-order condition for the optimum are met. The public operator forces the private operator to set optimum tolls.

## A.7 Proof of Corollary 2

Follows by setting partial derivatives to zero in (9).

## B Discussion of proposition 2

This result is better illustrated in a simplified network with two links between a given origin-destination pair. Let the free-flow travel cost on the longer link be fixed to  $f_0$ ,<sup>4</sup> the free-flow travel cost on the shorter link be  $f + a n$  where  $f$  and  $a$  are given positive numbers and  $n$  the number of the users of the shorter link. An interior solution requires  $f < f_0 < f + a N$ , a condition that we assume. At equilibrium, the user costs are equal on the roads and a simple algebra shows that  $n^e = (f_0 - f)/a$ . The total number of the users who travel from the origin to the destination is  $N$  (inelastic demand), and we denote the users of the shorter link  $n$ , so  $N - n$  is the number of the longer road. At equilibrium, it is clear that the total cost is  $n(f + a n) + f_0(N - n)$  which reaches its minimum at  $n^O = (f_0 - f)/2a$ . A road toll  $\tau^O = (f_0 - f)/2$  impose on the users of the shorter route will decentralize the optimum. At the same time, if that road is managed by a private operator who maximizes her profit  $\tau n$  under the constraint that user costs are equal, then he will maximize his profit by imposing the optimum toll, i.e.  $\tau^{sp} = \tau^O = (f_0 - f)/2$ .

So, we can state that the road toll imposed by the private operator is the sum of two parts. The first one is the optimum toll that would have been imposed by a public manager, and the second part correspond mainly on crowding in transit. The second part is zero if there is no crowding and increases as the impact of crowding increases. Analytically, we write the road toll for link  $(ij)$  as

$$\tau_{ij}^{sp} = \tau_{ij}^O + (\text{crowding-induced part}).$$

The existence of crowding in public transit reduces the magnitude of the demand elasticity of the users of the road. The private operator, benefits from this rigidity in the users choice, by increasing the road toll.

In the linear case, and focusing on an interior solution, we can obtain the tolls imposed by the private operator. Since the derivative with respect to the number of users is constant we can easily express these tolls as function of the first-best tolls. Doing so, we

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<sup>4</sup>The notation is specific to this example.

obtain

$$\begin{aligned}\tau_{sc}^{sp} &= \tau_{sc}^O + \frac{1}{2} \left[ p_{sc} + a_{sc}^T \frac{N_{sc} + N_{se}}{f_{sc}} \right] \\ \tau_{ce}^{sp} &= \tau_{ce}^O + \frac{1}{2} \left[ p_{ce} + a_{ce}^T \frac{N_{ce} + N_{se}}{f_{ce}} \right] \\ \tau_{se}^{sp} &= \tau_{sc}^{sp} + \tau_{ce}^{sp} + \frac{1}{2} \left[ c_s + (\alpha - 1) \frac{c_w}{2} + (\beta - 1) p_{ce} + F_{ce}^R + F_{sc}^R - F_{se}^R \right].\end{aligned}$$

So, for the same fares,  $\tau_{ij}^{sp} > \tau_{ij}^O$  for all  $(ij)$ . Roads are underused and public transport is overused induced large crowding levels. As a response, the public operator of transit will reduce total travel cost by imposing positive fares. Comparing with the first-best optimum considered above and where transit fares were fixed to zero, we find that second-best fares when public transport is operated by a private operator are higher than first-best fares. The case where  $N_{sc} = N_{ce}$  provide a particularly simple expressions, and the public operator can reach the first-best mode choice by imposing fares given by

$$\begin{aligned}p_{ce}^O &= \frac{N_{sc} + N_{ce}}{\frac{1-\beta}{a_{sc}^R} + \frac{f_{ce}}{a_{ce}^T}} \\ p_{sc}^O &= \frac{f_{ce} a_{sc}^T}{f_{sc} a_{ce}^T} p_{ce}^O\end{aligned}$$

Notice that this result is in line with conclusions of Kilani et al. (2014), Kraus (2012) to the case of the model studied here.