Intertemporal equilibrium with heterogeneous agents, endogenous dividends and collateral constraints*

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Abstract

We build a dynamic general equilibrium model with heterogeneous producers and financial market imperfections (collateral constraints and incompleteness). First, we prove the existence of equilibrium and provide a tractable characterization to check whether a sequence is an equilibrium. Second, we study the effects of financial imperfections on economic growth and land prices. Third, we develop a theory of valuation of land by introducing the notion of endogenous land dividends (or yields) and different concepts of land-price bubbles. Some examples of bubbles are provided in economies with and without short-sales.

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1 Introduction

The interplay between asset prices and economic activities is an important topic, especially after the Great Recession. A vast literature has flourished on these transmission mechanisms focusing on the key notion of asset-price bubble. Many articles have addressed this issue in overlapping generations (OLG) models (Tirole, 1985; Farhi and Tirole, 2012; Martin and Ventura, 2012), and others have adopted an infinite-lived agent's approach (Tirole, 1982; Santos and Woodford, 1997; Kocherlakota, 2009; Hirano and Yanagawa, 2016). However, most of this literature ignores the productive role of assets. Our paper aims to develop a theory of asset valuation in the case the asset is not only a collateral but also an input. We contribute to explain the asset pricing in terms of production activity. Although many papers have raised the question of asset valuation, most of them have focused on assets with exogenous positive dividends (Lucas, 1978; Santos and Woodford, 1997) or zero dividends (Bewley, 1980; Tirole, 1985; Pascoa et al., 2011). Unlike this literature, in our paper, every

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agent can use the asset to produce the consumption good according to her own technology. More precisely, we consider an infinite-horizon general equilibrium model with three assets: a consumption good, land to produce this good, and a financial asset with zero supply. There is a finite number of agents who differ in terms of endowments, technology, preferences and borrowing limits. In each period, any agent produces, exchanges and consumes. In the spirit of Geanakoplos and Zame (2002) and Kiyotaki and Moore (1997), agents can borrow but must hold land as collateral. The repayment does not exceed a given fraction of land income because there is lack of commitment.[1]

Before exploring the equilibrium properties, we prove the existence of equilibrium in stochastic production economies with borrowing constraints and incomplete markets. There is a vast literature on this issue. We only mention here some papers that are very closed to ours.[2] Geanakoplos and Zame (2002) prove the existence of collateral equilibrium in a two-period models with durable goods and collateralized securities. By extending Geanakoplos and Zame (2002), Araujo et al. (2002) prove the existence of equilibrium in an infinite-horizon model with a collateral requirement on the sale of financial assets. Kubler and Schmedders (2003) construct and prove, with additional conditions on the exogenous variables, the existence of a Markov equilibrium in an infinite-horizon asset pricing model with incomplete markets and collateral constraints; such a Markov equilibrium is also proved to be competitive equilibrium. Nevertheless, it remains difficult to study the equilibrium in models with incomplete markets: Gottardi and Kubler (2015) provide a tractable model with collateral constraints and complete markets. However, these papers did not take into account the production sector. Becker et al. (2015) prove the existence of a Ramsey equilibrium with endogenous labor supply and borrowing constraint on physical capital. Le Van and Pham (2016) extend Becker et al. (2015) by introducing a long-lived asset and short-sales. A proof à la Becker et al. (2015) or Le Van and Pham (2016) no longer applies in our model because agents trade short-lived financial assets with zero supply instead of long-lived assets. The challenge is to prove that individual asset volumes are bounded. To overcome the difficulty, we introduce an intermediate economy where the real asset is replaced by a nominal one. In this economy, we can bound the volume of financial asset and, therefore, adapt the proof of existence à la Becker et al. (2015). We end up by building an equilibrium in the original economy from the one in the intermediate economy. Our proof can apply to a large class of general equilibrium models used in macroeconomics. It contributes to the novelty of the paper. Moreover, we provide a tractable characterization to check whether a sequence is an equilibrium or not. In our paper, transversality conditions are proved, but not imposed.

The general equilibrium perspective is suitable to represent the interference between financial markets and economic activities. Some of equilibrium properties deserve mention.

(1) If any agent produces (this happens under Inada's condition), then agents' marginal productivities turn out to be the same. Cases where some agents give up the production are quite specific: they experience a very low productivity while the others have high productivity and a full access to credit markets. However, it should be noticed that when financial markets are incomplete or agents are prevented from borrowing, the most productive agent may not buy land to produce. We provide examples illustrating this idea.

(2) The steady state analysis is carried out in a simple case.[3] In the long run, the most patient agent may not hold the entire stock of land. This result challenges the well-known Ramsey's conjecture (the most patient individual owns all the capital in the long run) and

[1]The land income here is the sum of the value of land and its fruit.

[2]An excellent introduction to asset pricing models with incomplete markets and infinite-horizon can be found in Magill and Quinzii (2008)

[3]The steady state may not exist when endowments are not stationary.

looks more realistic. The very reason is that, in our model, any agent is a producer differently from what happens in the growth literature where consumers rent capital to a representative firm (see Becker and Mitra (2012) among others).

The last part of the paper develops a theory of land valuation. We focus on rational land-price bubbles and their economic consequences.[4] The standard literature (Lucas, 1978; Santos and Woodford, 1997) considers long-lived assets paying exogenous dividends. In our model, any agent may produce with a landowner-specific technology which may be non-linear: this prevents the standard approach from applying. Now, one unit of land yields an endogenous amount of consumption good in any future state, the so-called *dividend* or *added-value of land*. In deterministic economies,[5] (γ_t, d_t) is called a process of state-prices and land dividends if the the following intertemporal asset-pricing equation holds:

$$q_t = \gamma_{t+1} (q_{t+1} + d_{t+1}) \tag{1}$$

jointly with the condition that the dividend d_{t+1} is higher than the lowest marginal productivity of agents (every agent is allowed to produce). Here, q_t is the land price in terms of consumption good. Equation (1) is a no-arbitrage condition: what we pay today to buy 1 unit of land is equal to what we will receive by reselling 1 unit of land plus land dividends (in terms of consumption good). Of course, when agents share the same linear technology, we recover the Lucas' tree with exogenous dividends. Santos and Woodford (1997) and Montrucchio (2004) also use no-arbitrage conditions to define state-price process. They use them to evaluate assets having exogenous dividends in economies with incomplete financial markets.

In our economy, land plays three different roles: once we buy land, we can (1) resell it, (2) use it to produce or (3) to borrow (collateral role). The land dividend represents the two last roles. According to the literature (Tirole, 1982; Santos and Woodford, 1997), the fundamental value of land is defined at equilibrium as the sum of the discounted values of such endogenous dividends and the bubble is the (positive) difference between the price of land and its fundamental value. Our approach is suitable not only for deterministic but also for stochastic economies.

Our general result indicates that when agents are uniformly impatient, for any process of state-price and dividends such that the discounted value of aggregate consumption good vanishes at the infinity, the price of land equals its fundamental value. This is an extension of Theorem 3.3 in Santos and Woodford (1997)[6] to the case of productive assets. We also prove that, under uniform impatience in bounded economies, bubbles are ruled out for any process of state-price and dividends.

The assumption of uniform impatience can be removed in the case of deterministic economies. In such a case, we find out some interesting features. Let us mention two of them. First, land-price bubbles arise only if the borrowing constraints of some agent are binding infinitely many times.[7] Indeed, when the borrowing constraints are not binding, the marginal rates of substitution of all agents are the same and, hence, the no-bubble condition

[4]We refer to Miao (2014) for an introduction to bubbles in infinite-horizon models. A survey on bubbles with asymmetric information, overlapping generations, heterogeneous beliefs can be found in Brunnermeier and Oehmke (2012).

[5]For the sake of simplicity, we mention here only the deterministic case. The stochastic one is presented in Section 6.

[6]Theorem 3.3 in Santos and Woodford (1997) shows that, under uniform impatience, assets in positive net supply and exogenous dividend have no bubble for state-prices that yield finite present values of aggregate endowments

[7]This is only a necessary condition for the existence of bubble. This issue will be addressed in Sections 5.1.1 and 5.2.

becomes equivalent to a no-Ponzi scheme. Since the transversality conditions are satisfied, the no-bubble condition is verified as well. Second, there is always one agent whose expected value of land equals the land price.

We also provide some new examples of bubbles in economies with and without short-sales. In the first one, agents are prevented from borrowing (with zero borrowing limits), their endowments fluctuate and/or technologies are non-stationary. Agents face a drop in endowments at time t, but are unable to borrow and transfer wealth from period $t+1$ to period t. Thus, they can buy land at date $t-1$ at a higher price, independent on their technologies, in order to transfer their wealth from date $t-1$ to date t. When the agents' TFP goes to zero, the fundamental value of land tends to zero as well, below the above level of price. By consequence, a land bubble arises. In the second example, short-sales are allowed but borrowing constraints still work. As in the first example, at any date, at least one agent is forced to save, hence she may accept to buy land at a higher price or to buy a financial asset with low interest rates. Therefore, bubbles may occur.

Our concept of land bubble contributes to the literature on bubbles of assets delivering endogenous dividends. Among others, three approaches deserve mention.

Araujo et al. (2011), Pascoa et al. (2011) introduce and study concepts of bubbles of durable goods, collateralized assets and fiat money. They provide asset-pricing conditions based on the existence of so-called deflators and non-pecuniary returns; then, they use them to defined bubbles. In the current paper, we focus on land and non-linear production functions.

Miao and Wang (2012, 2015) consider bubbles on the firm's value with endogenous dividends. They split this value in two parts: $V(K) = \mathcal{Q}K + B$, where K is the initial stock of the firm, $\mathcal{Q}$ is the marginal Tobin's Q ($\mathcal{Q}$ is endogenous) and B is the bubble. They interpret $\mathcal{Q}$ as the fundamental value of firm. This approach, which is different from ours, cannot be directly applied for valuation of land in our model because land is used by many agents while a stock in Miao and Wang (2012, 2015) is issued by only one firm and stock dividends are taken as given by other agents.

Becker et al. (2015), Bosi et al. (2015) introduces the concept of physical capital bubble in one- and two-sector models. They define the fundamental value of physical capital as the sum of discounted values of capital returns (net of depreciation). As above, a physical capital bubble exists if the equilibrium price of physical capital exceeds this fundamental value. In our model, land, as input, looks like capital. The difference is that Becker et al. (2015) consider a representative firm while, in our work, each agent can be viewed as an entrepreneur.

The rest of the paper is organized as follows. Section 2 presents the framework and provides some preliminary equilibrium properties. Sections 3 and 4 study the role of financial market imperfections and land bubbles. Section 5 presents examples of bubbles in deterministic economies with and without short-sales. Section 6 extends our analysis to the stochastic case. Section 7 concludes. Technical proofs are gathered in the appendices.

2 Framework

We firstly consider an infinite-horizon general equilibrium model without uncertainty. A model with uncertainty will be presented in Section 6. The time is discrete and runs from date 0 to infinity. The number m of agents is finite. I denotes the set of agents.

Consumption good. There is a single consumption good. At each period $t = 0, 1, 2, \ldots$, the price of consumption good is denoted by p_t. Each agent i is endowed with $e_{i,t}$ units of consumption good, and chooses to consume $c_{i,t}$ units of this good.

Land. We denote by L and q_t the (exogenous) total supply of land and its price at date t. At this date, the agent i buys $l_{i,t+1}$ units of land. More precisely, (1) she uses this land to produce $F_i(l_{i,t+1})$ units of consumption good and (2) she resells land at a price q_{t+1}. F_i is the production function of agent i.

The financial market opens at the initial date. If agent i buys $a_{i,t}$ units of financial asset at date $t-1$, she will receive $R_t a_{i,t}$ (in terms of money) at date t, where R_t is the gross return. Agents can also borrow in the credit market. However, if they do that, they are required to hold land as collateral. The sense and the role of borrowing constraints will be explained below.

Each household i takes the sequence of prices $(p, q, R) := (p_t, q_t, R_t)_{t=0}^{\infty}$ as given and chooses sequences of consumption, land, and asset volume $(c_i, l_i, a_i) := (c_{i,t}, l_{i,t}, a_{i,t})_{t=0}^{+\infty}$ in order to maximizes her intertemporal utility

$$P_i(p, q, R) : \quad \max_{(c_i, l_i, a_i)} \sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t})$$

subject to, for each t,

$$l_{i,t} \geq 0 \tag{2}$$
$$p_t c_{i,t} + q_t l_{i,t} + p_t a_{i,t} \leq p_t e_{i,t} + q_t l_{i,t-1} + p_t F_i(l_{i,t-1}) + R_t a_{i,t-1} \tag{3}$$
$$R_{t+1} a_{i,t} \geq -f_i \big[q_{t+1} l_{i,t} + p_{t+1} F_i(l_{i,t})\big] \tag{4}$$

where $l_{i,-1} > 0$ is given and $a_{i,-1} = 0$ (no debt before the opening of financial markets).

Borrowing constraint (4) means that agent i can borrow an amount whose repayment does not exceed an exogenous share of land income. The parameter f_i is set by law and below 1. Parameter f_i can be viewed as the borrowing limit of agent i.

Condition (4) can also be interpreted as a collateral constraint: agents can borrow but they must own land as collateral. In case of default, lenders can seize the fraction f_i of agent i's land income. Here, we assume $f_i \leq 1$ because there is lack of commitment. Parameter f_i captures possible looses associated with the reallocation of land in case of default (see Quadrini (2011) for a review of this issue). By the way, our model is also related to the literature on general equilibrium with collateral constraints (Geanakoplos and Zame, 2002; Kiyotaki and Moore, 1997; Kubler and Schmedders, 2003). However, it is different from Kiyotaki and Moore (1997) where the repayment does not exceed the revenue from reselling land. In other words, Kiyotaki and Moore (1997) require

$$R_{t+1} a_{i,t} \geq -q_{t+1} l_{i,t}.$$

In the case $f_i = 1$, constraint (4) above corresponds to solvency constraint (4) in Chien and Lustig (2010). Our model is also related to Liu et al. (2013) where land-price dynamics are considered.

Remark 1. *1. Kiyotaki and Moore (1997) consider two types of agents, a farmer and a gatherer, with different time preferences: $\beta < \beta'$. The farmer has a linear production function, while the gatherer has a decreasing return to scale production function. Kiyotaki and Moore (1997) look at the equilibrium properties around the steady state and assume that R_t/p_t is exogenous and equal to $1/\beta'$ and that $f_i = 1$ for every i. Contrary to Kiyotaki and Moore (1997), we do not require these assumptions and we will provide global analysis of intertemporal equilibria.*

2. *Our model differs also from Farhi and Tirole (2012). Indeed, we consider dynamic firms in an infinite-horizon GE model while they focus on firms living for* 3 *periods in an OLG model.*

The economy, denoted by $\mathcal{E}$, is characterized by a list of fundamentals

$$\mathcal{E} := (u_i, \beta_i, e_i, f_i, l_{i,-1}, a_{i,-1}, F_i).$$

Definition 1. *Given the economy* $\mathcal{E}$*. A list* $\left(\bar{p}_t, \bar{q}_t, \bar{R}_t, (\bar{c}_{i,t}, \bar{l}_{i,t}, \bar{a}_{i,t})_{i=1}^m\right)_{t=0}^{+\infty}$ *is an equilibrium if the following conditions are satisfied:*

(i) Price positivity: $\bar{p}_t, \bar{q}_t, \bar{R}_t > 0$ *for* $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\text{good: } \sum_{i=1}^m \bar{c}_{i,t} = \sum_{i=1}^m (e_{i,t} + F_i(\bar{l}_{i,t-1})) \tag{5}$$

$$\text{land: } \sum_{i=1}^m \bar{l}_{i,t} = L \tag{6}$$

$$\text{financial asset: } \sum_{i=1}^m \bar{a}_{i,t} = 0. \tag{7}$$

(iii) Agents' optimality: for each i, $(\bar{c}_{i,t}, \bar{l}_{i,t}, \bar{a}_{i,t})_{t=0}^{\infty}$ *is a solution of the problem* $P_i(\bar{p}, \bar{q}, \bar{R})$.

Note that the financial asset in our framework is a short-lived asset with zero supply, which is different from the long-lived asset bringing exogenous positive dividends in Lucas (1978), Kocherlakota (1992), Santos and Woodford (1997), Le Van and Pham (2016). Instead, when production functions are not stationary and given by $F_{i,t}(x) = d_t x$, land in our model corresponds to this asset with exogenous dividends; in particular, when $F_i = 0$ for every i, land becomes fiat money as in Bewley (1980) or pure bubble asset as in Tirole (1985).

2.1 The existence of equilibrium

In what follows, if we do not explicitly mention, we will work under the following assumptions.

Assumption 1 (production functions)**.** *For each* i*, the function* F_i *is concave, continuously differentiable,* $F_i' > 0$ *and* $F_i(0) = 0$.

Notice that we consider both cases $F_i'(0) = \infty$ and $F_i'(0) < \infty$ (linear production functions satisfy this condition).

Assumption 2 (endowments)**.** $l_{i,-1} > 0, a_{i,-1} = 0$ *for any* i*, and* $e_{i,t} > 0$ *for any* i *and for any* t.

Assumption 3 (borrowing limits)**.** $f_i \in (0, 1]$ *for any* i.

Assumption 4 (utility functions)**.** *For each* i*, the function* u_i *is continuously differentiable, concave,* $u_i(0) = 0$*,* $u_i(+\infty) = +\infty$*,* $u_i' > 0$*,* $u_i'(0) = \infty$.[8]

[8] In the proof of the existence of equilibrium, we do not require $u_i'(0) = \infty$. This condition is to ensure that $c_{i,t} > 0$ for any t, which is used in the rest of the paper.

Assumption 5 (finite utility). *For each i*

$$\sum_{t=0}^{\infty} \beta_i^t u_i(W_t) < \infty, \text{ where } W_t := \sum_{i=1}^{m} \Big(e_{i,t} + F_i(L)\Big). \tag{8}$$

Proposition 1. *Under the above assumptions, there exists an equilibrium.*

A proof for this result is presented in the online appendix. Notice that in this proof, we allow for non-stationary production functions. However, in this paper (except Section 5 and Section 6), we assume that the technology is stationary for the sake of simplicity.

Price normalization: *Since $p_t > 0$ at equilibrium, we will normalize by setting $p_t = 1$ for any t for the rest of the paper.* In this case, an equilibrium is denoted $\Big(q, R, (c_i, l_i, a_i)_{i=1}^m\Big)$. We keep in mind that $q_t \in (0, \infty)$ and $R_t \in (0, \infty)$ are land price and asset return in terms of consumption good.

2.2 Borrowing constraints and transversality conditions

We provide the following fundamental result: a tractable necessary and sufficient condition to check whether a sequence is an intertemporal equilibrium or not.

Proposition 2.

1. *Let $\Big(q, R, (c_i, l_i, a_i)_{i=1}^m\Big)$ be an equilibrium. There exists a positive sequence of multipliers $(\lambda_{i,t}, \eta_{i,t}, \mu_{i,t+1})$ such that*

$$\textit{FOCs: } \beta_i^t u_i'(c_{i,t}) = \lambda_{i,t} \tag{9}$$

$$\lambda_{i,t} = (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} \tag{10}$$

$$\lambda_{i,t} q_t = (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + F_i'(l_{i,t})) + \eta_{i,t} \tag{11}$$

$$\eta_{i,t} l_{i,t} = 0 \tag{12}$$

$$\mu_{i,t+1}\Big(R_{t+1} a_{i,t} + f_i\big[q_{t+1} l_{i,t} + F_i(l_{i,t})\big]\Big) = 0 \tag{13}$$

$$\textit{Transversality condition: } \lim_{t\to\infty} \lambda_{i,t}(q_t l_{i,t} + a_{i,t}) = 0 \tag{14}$$

Moreover, we have, for any i,

$$\infty > \sum_{t=0}^{\infty} \lambda_{i,t} c_{i,t} = \lambda_{i,0}\big(F_i(l_{i,-1}) + q_0 l_{i,-1}\big) + \sum_{t=0}^{\infty} \lambda_{i,t} e_{i,t}$$
$$+ \sum_{t=1}^{\infty} \lambda_{i,t}\Big(1 + f_i \frac{\mu_{i,t}}{\lambda_{i,t}}\Big)\Big(F_i(l_{i,t-1}) - l_{i,t-1} F_i'(l_{i,t-1})\Big) \tag{15}$$

2. *If the sequences $(q, R, (c_i, a_i, l_i)_i)$ and $(\lambda_i, \eta_i, \mu_i)$ satisfy*

   (a) $c_{i,t}, l_{i,t}, \lambda_{i,t}, \eta_{i,t}, \mu_{i,t+1} \geq 0$; $p_t, q_t, R_t > 0$ *for any t;*

   (b) condition (3) is binding and conditions (2), (4), (5), (6), (7) hold for any t;

   (c) conditions (9), (10), (11), (12), (13), (14) hold for any t;

   (d) $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) < \infty$;

   then $(q, R, (c_i, a_i, l_i)_i)$ is an intertemporal equilibrium.

The challenge and key point of Proposition 2 is the necessity of transversality conditions (14). To prove this, we develop the method in Kamihigashi (2002). Recall that Kamihigashi (2002) only considers positive allocations while asset volume $a_{i,t}$ may be negative in our model. The detailed proof of this result is presented in Appendix 8.

Remark 2. *If we keep the notation p_t, transversality condition is* $\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t})(\frac{q_t}{p_t} l_{i,t} + a_{i,t}) = 0$.

According to (10), we have $\lambda_{i,t} \geq \lambda_{i,t+1} R_{t+1}$ for every i. Since $f_i > 0$ for any i, it is easy to see that there exists an agent i whose borrowing constraint (4) is not binding. Thus $\mu_{i,t+1} = 0$ which implies that $\lambda_{i,t} = \lambda_{i,t+1} R_{t+1}$. As a result, we get the following result:

Lemma 1. *We have*

$$\frac{1}{R_{t+1}} = \max_{i\in\{1,\dots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} \tag{16}$$

We define the discount factor γ_{t+1} ($\gamma_{i,t+1}$) of the economy (agent i) from date t to date $t+1$, and the discount factor Q_t ($Q_{i,t}$) of the economy (agent i) from the initial date to date t as follows

$$\gamma_{t+1} := \max_{i\in\{1,\dots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}, \quad Q_0 := 1, \quad Q_t := \gamma_1 \dots \gamma_t$$
$$\gamma_{i,t+1} := \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}, \quad Q_{i,0} := 1, \quad Q_{i,t} := \gamma_{i,1} \cdots \gamma_{i,t} = \frac{\beta_i^t u_i'(c_{i,t})}{u_i'(c_{i,0})}$$

Note that $\gamma_{i,t} \leq \gamma_t$ for any i and t.

We rewrite borrowing constraint (4) as

$$Q_{t+1} R_{t+1} a_{i,t} \geq -f_i Q_{t+1}\big[q_{t+1} l_{i,t} + F_i(l_{i,t})\big]$$

According to definition of (Q_t) and Lemma 1, we see that $Q_t = R_{t+1} Q_{t+1}$. Therefore, borrowing constraint (4) is equivalent to

$$Q_t a_{i,t} \geq -f_i Q_{t+1}\big[q_{t+1} l_{i,t} + F_i(l_{i,t})\big]$$

Corollary 1 (fluctuation of borrowing constraints)**.** *At equilibrium, we have:*

1. *For each i, there are only two cases:*
   (a) $\lim_{t\to\infty} \Big(Q_t a_{i,t} + f_i Q_{t+1}\big(q_{t+1} l_{i,t} + F_i(l_{i,t})\big)\Big)$ *does not exist;*
   (b) $\lim_{t\to\infty} \Big(Q_t a_{i,t} + f_i Q_{t+1}\big(q_{t+1} l_{i,t} + F_i(l_{i,t})\big)\Big) = 0$.
2. *(transversality condition, version 2) We have, for each i,*

$$\liminf_{t\to\infty} \Big(Q_t a_{i,t} + f_i Q_{t+1}\big(q_{t+1} l_{i,t} + F_i(l_{i,t})\big)\Big) = 0 \tag{17}$$

We observe that there are two kinds of transversality conditions. The first one is (14) which is determined by the individual discount factor $\beta_i^t u_i'(c_{i,t})/u_i'(c_{i,0})$. It characterizes the optimality of agent i's allocations. The second one is (17) based on the economy discount factor Q_t. It clarifies the role of borrowing constraints.

Remark 3. *All the results in this section apply also to non-stationary production functions.*

3 The role of the financial market

For each $t \geq 1$, we introduce two productive bounds:

$$\underline{d}_t := \min_{i \in \{1,\ldots,m\}} F_i'(l_{i,t-1}), \quad \bar{d}_t := \max_{i \in \{1,\ldots,m\}} F_i'(l_{i,t-1}).$$

We have the following result showing the relationship among land prices, marginal productivities, interest rates and borrowing limits.

Lemma 2. *The relative price of land is governed by the following inequalities:*

$$\gamma_{t+1}\big(q_{t+1} + \underline{d}_{t+1}\big) \leq q_t \leq \gamma_{t+1}\big(q_{t+1} + \bar{d}_{t+1}\big) \quad (18)$$

$$f_i\gamma_{t+1}\big(q_{t+1} + F_i'(l_{i,t})\big) \leq q_t \quad (19)$$

for any i and t.

According to (18), we introduce the land dividends (or value added of land, or land yields).

Definition 2 (dividends of land). *The dividends of land $(d_t)_t$ is defined by the following no-arbitrage condition*

$$q_t = \gamma_{t+1}\big(q_{t+1} + d_{t+1}\big) \quad (20)$$

Interpretation. Once we buy land, we will be able to resell land and expect to receive an amount. This amount is exactly the dividend of land defined by (20). Equation (20) is a no-arbitrage condition: what we pay to buy 1 unit of land at date t is equal to what we receive by reselling 1 unit of land plus the dividend of land (in terms of consumption good). When technologies are linear and identical ($F_i(X) = dX$ for any i), we have $d_t = d$ for any t, and hence we recover the Lucas' tree. In our general setup, (18) implies that any land dividend (d_t) is greater than the lowest marginal productivity ($\underline{d}_t$) but less than the highest one ($\bar{d}_t$).

In our model, land has a threefold structure: after buying land at date t, agents (1) resell it at date $t+1$, (2) use it as collateral in order to borrow from financial markets and (3) receive an amount of consumption good from their production process. Definition 2 states that dividends are endogenous and capture the roles (2) and (3) of land. In fact, because land is resold and gives dividends at each date, (20) can be interpreted as an asset-pricing or a no-arbitrage condition.

We point out some interesting properties of land dividends.

Lemma 3 (fair financial system). *$d_{t+1} = \bar{d}_{t+1}$ if $f_i = 1$ for any i or (4) is not binding for any i.*

We can interpret $f_i = 1$ as a full access to credit market for agent i. Lemma 3 points out that the land dividend equals the highest marginal productivity if either anyone may fully enter the credit market or borrowing constraints of any agent are not binding.

This following result shows that dividends equal the lowest marginal productivities if every agent buys land.

Proposition 3. *Focus on date t and assume $l_{i,t-1} > 0$ for every i. In this case, $d_t = \underline{d}_t$.*

We highlight some consequences of Lemma 3 and Proposition 3.

Corollary 2. *1. If $F_i'(0) = +\infty$ for every i, then $d_t = \underline{d}_t$ $\forall t$.*

2. *(equal marginal productivities.) If* $\mu_{i,t} = f_i\mu_{i,t}$[9] *and* $l_{i,t-1} > 0$*, then* $F_i'(l_{i,t-1}) = d_t$ $\forall i$.

3. *If* $f_i = 1$ *and* $F_i'(0) = \infty$ *for any* i*, then* $d_t = F_i'(l_{i,t-1})$ $\forall i, t$.

3.1 Who buys land? Who needs credits?

In this section, we will find conditions under which agents become producers and/or borrowers. First, land demand depends on its productivity.

Proposition 4. *If* $l_{i,t} > 0$ *then* $F_i'(l_{i,t}) \geq d_{t+1}$.
If $F_i'(l_{i,t}) > d_{t+1}$*, then borrowing constraint (4) of agent* i *is binding.*

The first statement means that if an agent buys land, its marginal productivity must be greater than land dividends.

The second one shows that if an agent has a marginal productivity which is strictly greater than land dividend, she will borrow until her borrowing constraints become binding. In other words, this agent needs credit.

The next result suggests that agents with a low productivity do not buy land to produce.

Proposition 5. *Focus on agent* i *and assume that there exists an agent* j *such that* $f_j = 1$ *and* $F_i'(0) < F_j'(L)$*. We have* $l_{i,t} = 0$ *for every* t.

Notice that Proposition 5 holds whatever the form of utility functions and the size of the discount rate β_i.

We can interpret $f_j = 1$ as a full access of agent j to credit market. In this case, any agent i with lower productivity ($F_i'(0) < F_j'(L)$) never produces. Proposition 5 is in line with Proposition 1 in Le Van and Pham (2016) where they prove that nobody invests in the productive sector if the productivity of this sector is too low.

Agents can be reinterpreted as countries. In this case, our economy works as a world economy with free trade. Each country i is endowed with $l_{i,0}$ units of land. When the trade is fully free and the international financial market is good enough (in the sense that $f_i = 1$ for any i), countries with a lower productivity never produce and land in these countries will be held by the countries with the highest productivity.

Remark 4. *When there exist financial (or political) frictions characterized by* $f_j < 1$*, the analysis becomes more complex. In Section 5.1.1, we will present an example where there are two agents: A and B with* $f_A = 0$ *(agent A is prevented from borrowing) with linear technologies. In this example, at date* $2t + 1$*, the productivity of agent A is higher than that of agent B, but agent B may produce at date* $2t + 1$.

3.2 A particular case: a steady state analysis

In this section, we assume that agents have no endowments, that is $e_{i,t} = 0$ for every i and t. For simplicity, we also assume that there are two agents, say i and j, with different rates of time preference: $\beta_i < \beta_j$.

We give an analysis at the steady state. Recall that when endowments are not stationary, the existence of steady state may not hold.

[9] This condition is satisfied if $f^i = 1$ or $R_t a_{i,t-1} > -f_i\big[q_t l_{i,t-1} + F_i(l_{i,t-1})\big]$, i.e, the borrowing constraint of agent i at date t is not binding.

Lemma 4. *Consider two agents i and j with $\beta_i < \beta_j$. If $e_{i,t} = 0$ for any i and t, and $F_i(l_i) = A_i l_i^{\alpha}$, where $\alpha \in (0,1)$, for any i, then there is a unique steady state (up to a scalar for prices):*

$$R = \frac{1}{\beta_j} \tag{21}$$

$$q^{\frac{1}{1-\alpha}} L = \Big(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1} \Big)^{\frac{1}{1-\alpha}} + \Big(\frac{\alpha A_j}{\frac{1}{\beta_j} - 1} \Big)^{\frac{1}{1-\alpha}} \tag{22}$$

$$l_i = \Big(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1} \frac{1}{q} \Big)^{\frac{1}{1-\alpha}}, \quad l_j = L - l_i \tag{23}$$

$$Ra_i + f_i\big[ql_i + F_i(l_i)\big] = 0, \quad a_i + a_j = 0. \tag{24}$$

Who will own land in the long run?

Cobb-Douglas technologies imply $l_i, l_j > 0$. Each agent holds a strictly positive amount of land to produce themselves. In this respect, our model differs from Becker and Mitra (2012) where the most patient agent holds the entire stock of capital in the long run. The difference rests on two reasons.

First, in Becker and Mitra (2012), the firm is unique and consumers do not produce. In our model, any agent produces with her own technology and can be viewed as a credit-constrained entrepreneur.

Second, in Becker and Mitra (2012), returns on capital are determined by the marginal productivity of their representative firm. In our framework, land dividends are interpreted as land returns and determined by no-arbitrage condition (20).

Corollary 3 (the role of borrowing limit)**.** *Under conditions in Lemma 4, we have:*

1. *The relative price of land q increases in f_i.*
2. *The long-run quantity of fruits, i.e., $Y := F_i(l_i) + F_j(l_j)$ is increasing in the borrowing limit f_i.*

The intuition of point 1 is that when f_i increases, agent i can borrow more and, then, land demand increases in turn raising the price of land at the end.

The point 2 is also intuitive: the higher the level of f_i, the more the quantity the agent with the highest productivity can borrow, and, finally, the more the output produced.

4 Land bubbles

Combining $Q_{t+1} = \gamma_{t+1} Q_t$ with (20), we get

$$Q_t q_t = Q_{t+1}\big(q_{t+1} + d_{t+1}\big) \tag{25}$$

and, so,

$$\begin{aligned} q_0 &= \gamma_1(q_1 + d_1) = Q_1 d_1 + Q_1 q_1 = Q_1 d_1 + Q_1 \gamma_2 (q_2 + d_2) = Q_1 d_1 + Q_2 d_2 + Q_2 q_2 \\ &= \ldots = \sum_{t=1}^{T} Q_t d_t + Q_T q_T \end{aligned} \tag{26}$$

for any $T \geq 1$. This leads the following definition.

Definition 3 (bubble). *The fundamental value of land is defined by* $FV_0 := \sum_{t=1}^{\infty} Q_t d_t$. *We say that land bubbles exists if the market price of land (in term of consumption good) exceeds its fundamental value:* $q_0 > FV_0$.

As seen above, land dividends capture a twofold role of land: land is used to produce a consumption good and, as collateral, to borrow. The fundamental value of land reflects the value of both these roles.

As in Montrucchio (2004), Le Van and Pham (2014), some equivalences hold.

Proposition 6 (Necessary and sufficient conditions for bubbles). *The following statements are equivalent.*

(i) *Land bubbles exist.*

(ii) $\lim_{t\to\infty} Q_t q_t > 0$.

(iii) $\sum_{t=1}^{\infty} (d_t/q_t) < +\infty$.

Note that this result only depends on the no-arbitrage condition (25). Proposition 6 holds for any form of technologies, even non-stationary.

Since we are assuming that technologies are stationary, we have $d_t \geq \min_i F_i'(l_{i,t-1}) \geq \min_i F_i'(L) > 0$ for every t. This leads the following result.

Corollary 4. *If land bubbles exist, then* $\sum_{t=1}^{\infty} (1/q_t) < +\infty$.

This explains why the existence of land bubbles implies that real land prices tend to infinity. Notice, however, that this fact only holds in the case of stationary technology. In Section 5.1.1, this issue will be readdressed.

Interest rates and bubbles. According to (26), we have $\sum_{t=1}^{\infty} Q_t d_t \leq q_0 < \infty$ and, also, $d_t \geq \min_i F_i'(l_{i,t-1}) \geq \min_i F_i'(L) > 0$ for every t. Eventually, we get $\sum_{t=1}^{\infty} Q_t < \infty$.

We introduce the real interest rate of the economy ρ_t at date t as follows: $\gamma_t = 1/R_t = 1/(1+\rho_t)$. We notice that ρ_t may be negative. The condition $\sum_{t=1}^{\infty} Q_t < \infty$ writes explicitly

$$\sum_{t=0}^{\infty} \frac{1}{\prod_{s=1}^{t} (1+\rho_s)} < \infty$$

and we can reinterpret it by saying that the real interest rates are not "too low". We also observe that there exists a sequence of dates $(t_n)_n$ such that $\rho_{t_n} > 0$ for any n.

According to Proposition 6, land bubbles exist if and only if $\lim_{t\to\infty} \frac{1}{\prod_{s=1}^{t}(1+\rho_s)} q_t > 0$. This condition implies in turn $\lim_{t\to\infty} \frac{q_{t+1}}{q_t} \frac{1}{1+\rho_{t+1}} = 1$.

Hence, in the long run, if land bubbles exist, the rate of growth of land prices is equal to the gross interest rate.

4.1 No-bubble results

Proposition 7. *If* $Q_t/Q_{i,t}$ *is uniformly bounded from above for any* i, *then there are no bubbles.*

Write $\gamma_{i,t} = 1/(1+\rho_{i,t})$, where $\rho_{i,t}$ is interpreted as the the real expected interest rate of agent i at date t. As above, this interest rate may be negative. According to Proposition 7, if a bubble exists, there is an agent i such that her expected interest rates are high with respect to those of the economy in the following sense:

$$\prod_{s=1}^{T} \frac{1+\rho_{i,t}}{1+\rho_t} \xrightarrow{T\to\infty} \infty$$

Let us point out some consequences of Proposition 7.

Corollary 5. *If there exists $T > 0$ such that $\mu_{i,t} = 0$ for any i and $t \geq T$, then there is no land bubble.*

The intuition of this result is that when $\mu_{i,t} = 0$ for any i and $t \geq T$, the individual discount factors coincide with the discount factors of the economy. In this case, the no-bubble condition turns out to be equivalent to the no-Ponzi scheme. Since the transversality conditions are satisfied, the no-bubble condition holds as well.

Corollary 5 implies that if the borrowing constraints of any agent are not binding, then, there is no bubble. The following corollary clarifies it in other words.

Corollary 6 (bubble existence and borrowing constraints)**.** *If land bubbles exist, there exist an agent i and an infinite sequence of dates $(t_n)_n$ such that the borrowing constraints of agent i are binding at each date t_n, that is, for any t_n,*

$$R_{t_n} a_{i,t_n-1} = -f_i\big[q_{t_n} l_{i,t_n-1} + F_i(l_{i,t_n-1})\big]$$

Remark 5. *The binding of borrowing constraints is only a necessary condition for the existence of bubble. Sections 5.1.1 and 5.2 provide some examples where borrowing constraints of any agent are frequently binding but bubbles may not exist.*

The relationship between the existence of bubble and borrowing constraints are questioned in Kocherlakota (1992). He considers the borrowing constraints: $x_{i,t} \geq x$, where $x_{i,t}$ is the asset quantity held by agent i at date t and $x \leq 0$ is an exogenous bound. He claims that $\liminf_{t\to\infty}(x_{i,t} - x) = 0$ and interprets it concluding that borrowing constraints of agent i are frequently binding. He did not proved that $x_{i,t} - x = 0$ frequently.

We define the aggregate output of the economy at date t as $Y_t := \sum_{i=1}^{m} [e_{i,t} + F_i(l_{i,t-1})]$ and the present value of the aggregate output as $\sum_{t=0}^{\infty} Q_t Y_t$.

The main result of the section rests on the following list of four lemmas whose proofs are gathered in Appendix 10.

Lemma 5. *If $\sup_{i,t} e_{i,t} < \infty$ and technologies are stationary, the present value of the aggregate output is finite.*

Lemma 6. *Assume that $\sup_{i,t} e_{i,t} < \infty$ and technologies be stationary. Given an equilibrium, we obtain that $Q_t\big(l_{i,t}q_t + a_{i,t}\big)$ is uniformly bounded from below and from above as well.*

Lemma 7. *Let $\sup_{i,t} e_{i,t} < \infty$ and technologies be stationary. Given an equilibrium, the following limits exist:*

$$\lim_{t\to\infty} Q_t\big(q_t l_{i,t} + a_{i,t}\big) = \lim_{t\to\infty} \big(Q_t q_t l_{i,t-1} + Q_{t-1} a_{i,t-1}\big) \ \forall i. \tag{27}$$

Lemma 8. *Let* $\sup_{i,t} e_{i,t} < \infty$ *and technologies be stationary. Given an equilibrium, if there exists* T *such that*

$$f_i\Big(q_t + \frac{F_i(l_{i,t-1})}{l_{i,t-1}}\Big)l_{i,t-1} \geq (q_t + d_t)l_{i,t-1} \; \forall t \geq T, \tag{28}$$

then $\lim_{t\to\infty} Q_t\big(q_t l_{i,t} + a_{i,t}\big) \leq 0$.

Let us now state the main result of this section.

Proposition 8. *Assume that* $\sup_{i,t} e_{i,t} < \infty$ *and and* $f_i = 1$ *for every* i*. We also assume that all technologies be stationary and not zero. Then, land bubbles are ruled out at equilibrium.*

This proposition points out that there is no land bubble at equilibrium when the financial system is good enough (in the sense that $f_i = 1$ for any i), exogenous endowments are bounded from above and the technology is stationary.

Proposition 8 suggests that land bubbles only appear when land TFP of land technologies tends to zero and/or endowments grow without bound and/or agents cannot easily enter the financial market ($f_i < 1$). We will present some examples of bubbles in Section 5, where these conditions are violated.

Proposition 8 is in line with the results in Kocherlakota (1992), Santos and Woodford (1997), Huang and Werner (2000) and Le Van and Pham (2014), where they prove that bubbles are ruled out if the present value of aggregate endowments is finite. Indeed, the asset in Kocherlakota (1992) is a particular case of land in our model when $F_{i,t}(X) = \xi_t X$ for any i, X. Proposition 8 also shows that land bubbles are ruled out in Kiyotaki and Moore (1997).

Remark 6.

- *Proposition 8 still holds for any technology in the form* $A_{i,t}F_i$ *where* $A_{i,t}$ *is bounded away from zero for any* i*.*
- *Interestingly, although we are considering the utility function* $\sum_t \beta_i^t u_i(c_{i,t})$*, our proof of Proposition 8 still works in more general cases (for example, when the utility function takes the form* $\sum_t u_{i,t}(c_{i,t})$*).*[10]

4.2 Alternative concepts: individual and strong bubbles

According to (11), we have

$$\begin{aligned} q_t &= \frac{\lambda_{i,t+1} + f_i\mu_{i,t+1}}{\lambda_{i,t}}(q_{t+1} + F_i'(l_{i,t})) + \frac{\eta_{i,t}}{\lambda_{i,t}} \\ &= \frac{\lambda_{i,t+1}}{\lambda_{i,t}}\Big(q_{t+1} + \underbrace{F_i'(l_{i,t}) + \frac{\eta_{i,t}}{\lambda_{i,t+1}}}_{\text{Production return}} + \underbrace{\frac{f_i\mu_{i,t+1}}{\lambda_{i,t+1}}(q_{t+1} + F_i'(l_{i,t}))}_{\text{Collateral return}}\Big) \end{aligned}$$

We rewrite

$$q_t = \gamma_{i,t+1}\big(q_{t+1} + d_{i,t+1}\big) \tag{29}$$

and call $d_{i,t+1}$ *the individual dividend of agent* i *at date* $t+1$. Here $d_{i,t+1}$ includes two terms. The first one is $X_{i,t+1} := F_i'(l_{i,t}) + \dfrac{\eta_{i,t}}{\lambda_{i,t+1}}$ which represents the return from the production process.[11] The second term $\frac{f_i\mu_{i,t+1}}{\lambda_{i,t+1}}(q_{t+1} + F_i'(l_{i,t})$ can be interpreted as a collateral return.

[10] It should be noticed that our method here is no longer suitable for stochastic economies with incomplete markets. This issue will be addressed in Section 6.2.2.

[11] Note that $X_{i,t+1}l_{i,t} = F_i'(l_{i,t})l_{i,t}$.

Note that the collateral return is equal to zero if $f_i = 0$ or $\mu_{i,t+1} = 0$ (happen if borrowing constraint is not binding).

The asset-pricing equation (29) shows the way agent i evaluates the price of land. With the individual discount factor $\gamma_{i,t+1}$, once agent i buys land, she will be able to resell land at a price q_{t+1} and she will expect to receive $d_{i,t+1}$ units of consumption good as dividends. Since the individual discount factor $\gamma_{i,t+1}$ is less than that of economy γ_{t+1}, the individual dividend $d_{i,t+1}$ expected by agent i exceeds the dividend d_{t+1} of the economy.

Using (29) and adopting the same argument in (26), we find that, for each $T \geq 1$,

$$q_0 = \sum_{t=1}^{T} Q_{i,t} d_{i,t} + Q_{i,T} q_T$$

Definition 4 (individual bubble)**.** *1. $FV_i := \sum_{t=1}^{\infty} Q_{i,t} d_{i,t}$ is the i-fundamental value of land. We say that a i-land bubble exists if $q_0 > \sum_{t=1}^{\infty} Q_{i,t} d_{i,t}$.*

2. A strong bubble exists if the asset price exceeds any individual value of land, that is $q_0 > \max_i FV_i$ for any i.

The concept of $i-$bubble is closely related to bubbles of durable goods and collateralized assets in Araujo et al. (2011) or bubble of fiat money in Pascoa et al. (2011). Given an equilibrium, Araujo et al. (2011) provide asset-pricing conditions (Corollary 1, page 263) based on the existence of what they call *deflators* and *non-pecuniary returns* which are not necessarily unique. Then, they define bubbles associated to each deflators and non-pecuniary returns. In our framework, for each equilibrium, we give closed formulas for two types of deflators (we call γ_t and $\gamma_{i,t}$ discount factor and individual discount factor respectively). Unlike Araujo et al. (2011), the technology in our paper may be non-linear and non-stationary.

By applying the same argument in Proposition 6, we obtain some equivalences.

Proposition 9. *The following statements are equivalent.*

(i) i-land bubbles exist.

(ii) $\lim_{t\to\infty} Q_{i,t} q_t > 0$.

(iii) $\sum_{t=1}^{\infty} (d_{i,t}/q_t) < +\infty$.

Another added-value of our paper (comparing with Araujo et al. (2011), Pascoa et al. (2011)) is to study the connection between the concepts of bubble and i-bubble. This is showed in the following result.

Proposition 10. *1. Always $FV_0 \leq FV_i \leq q_0$ for any i. By consequence, if an i-land bubble exists for some agent i, then a land bubble exists.*

2. There is an agent i such that her i-bubble is ruled out. Consequently, strong land bubbles are ruled out, that is $q_0 = \max_i FV_i$.

3. If $FV_0 = FV_i$ for any i, then $FV_0 = FV_i = q_0$ for any i, that is, there is no room for bubbles nor i-bubble.

Comments and discussions. $FV_0 \leq FV_i \leq q_0$ follows from the definitions of bubble and i-bubble. The intuition is that, since any agent expects a higher interest rate than that of the economy, the individual value of land expected by any agent will exceed the fundamental value of land. Nevertheless, the converse of point 1 is not true. In Section 5.1.1, we present an example where i-bubble does not exist for any i while bubble may arise.

Points 2 shows that there is an agent whose expected value of land equals its equilibrium price. Point 3 is more intuitive and complements point 2: when any individual value of land coincides with that of economy, both bubble and individual bubbles are ruled out. However, when any individual value of land is identical but different from the fundamental value of land, we do not know whether land bubbles are ruled out.[12]

Our concept of strong bubble is related to the notion of *speculative bubble* in Werner (2014). He considers an asset bringing exogenous dividends in a model with ambiguity. Werner (2014) defines the asset fundamental value under the beliefs of agent i as the sum of discounted expected future dividends under her beliefs. He then says that speculative bubble exists if the asset price is strictly higher than any agent's fundamental value. The readers may ask why strong bubbles are ruled out while speculative bubbles in Werner (2014) may exist. It is hard to compare these two results since the two concepts of bubbles are defined in two different settings (with and without ambiguity).

Remark 7. *It should be noticed that Proposition 10 still holds in more general cases (for example, when the utility function is given by* $\sum_t u_{i,t}(c_{i,t})$*).*

5 Examples of bubbles

In this section, we contribute to the literature of bubbles by providing some examples where bubbles appear in deterministic economies even short-sales are allowed.[13] Here, dividends are endogenous determined and may be strictly positive.

5.1 Land bubbles without financial market

Focus on the case where there is no financial market. In this section, we allow for non-stationary production functions. Let us rewrite the agent's program. The household i takes the sequence of prices $(q) = (q_t)_{t=0}^{\infty}$ as given and chooses sequences of consumption and land $(c_i, l_i) := (c_{i,t}, l_{i,t})_{t=0}^{+\infty}$ in order to maximize her intertemporal utility

$$P_i(q): \quad \max \sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t}) \tag{30}$$

$$\text{subject to, for each } t,: \quad l_{i,t} \geq 0, \quad c_{i,t} + q_t l_{i,t} \leq e_{i,t} + q_t l_{i,t-1} + F_{i,t}(l_{i,t-1}), \tag{31}$$

where $l_{i,-1} > 0$ is given.

Under a linear technology ($F_{i,t}(x) = \xi_t x$ for every i), the land structure becomes the same asset structure as in Kocherlakota (1992), Santos and Woodford (1997) and Huang and Werner (2000). If $F_i = 0$ for every i, land becomes a pure bubble as in Tirole (1985).

Definition 5. *A list* $\left(\bar{q}_t, (\bar{c}_{i,t}, \bar{l}_{i,t})_{i=1}^m\right)_{t=0}^{+\infty}$ *is an equilibrium of the economy without financial market under the following conditions.*

[12] See observation "1. bubble vs i-bubble" in Section 5.1.1.

[13] Araujo et al. (2011) provide some examples of equilibria with bubbles in models where the utility functions take the form $\sum_{t\geq 0} \zeta_{i,t} u(c_{i,t}) + \epsilon_i \inf_{t\geq 0} u_i(c_{i,t})$. The parameter ϵ_i plays the key role.

(i) $\bar{q}_t \in (0, \infty)$ *for* $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\sum_{i=1}^{m} \bar{c}_{i,t} = \sum_{i=1}^{m} (e_{i,t} + F_{i,t}(\bar{l}_{i,t-1})), \quad \sum_{i=1}^{m} \bar{l}_{i,t} = L.$$

(iii) Agents' optimality: for each i, $(\bar{c}_{i,t}, \bar{l}_{i,t})_{t=0}^{\infty}$ *is a solution of the problem* $P_i(\bar{q})$.

Remark 8. *By applying Proposition 2 and Lemma 9 (Appendix 11), we can check that an equilibrium for the economy without financial market is a part of an equilibrium for the economy* $\mathcal{E}$ *with* $f_i = 0$ *for any* i.

Let $(q, (c_i, l_i)_{i=1}^m)$ be an equilibrium. Denoting by $\lambda_{i,t}$ and $\mu_{i,t}$ the multipliers associated to the budget constraint of agent i and to the borrowing constraint $l_{i,t} \geq 0$, we obtain the following FOCs for the economy $\mathcal{E}$:

$$\beta_i^t u_i'(c_{i,t}) = \lambda_{i,t}$$
$$\lambda_{i,t} q_t = \lambda_{i,t+1}(q_{t+1} + F'_{i,t+1}(l_{i,t})) + \eta_{i,t}, \quad \eta_{i,t} l_{i,t} = 0.$$

As above, we introduce the dividends of land: $q_t = \gamma_{t+1}\big(q_{t+1} + d_{t+1}\big)$, where γ_{t+1} is the discount factor of the economy from date t to date $t+1$: $\gamma_{t+1} := \max_{i \in \{1,\dots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}$.

We define the discount factor of the economy from initial date to date t as follows: $Q_0 := 1$ and $Q_t := \prod_{s=1}^{t} \gamma_s$ for any $t \geq 1$. Then, the fundamental value of the land is defined by $FV_0 := \sum_{t=1}^{\infty} Q_t d_t$. We say that land bubbles exist if $q_0 > FV_0$.

5.1.1 Examples of land bubbles

We now construct equilibria with bubbles.
The economy's fundamentals. Assume that there are two agents (A and B) with a common utility function $u_A(x) = u_B(x) = ln(x)$ but different non-stationary technologies $F_{A,t}(X) = A_t X, F_{B,t}(X) = B_t X$. For the sake of simplicity, we normalize the supply of land to one: $L = 1$, and we consider alternately null endowments: $e_{A,2t} = e_{B,2t+1} = 0$ for any t.

We need the following conditions to ensure the FOCs and identify the right sequence of discount factors of the economy (γ_t).

$$\beta_A\Big(\frac{\beta_B e_{B,2t}}{1+\beta_B} + A_{2t}\Big)\Big(\frac{\beta_A e_{A,2t+1}}{1+\beta_A} + A_{2t+1}\Big) \leq \beta_B \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t+1}}{1+\beta_A} \tag{32}$$

$$\beta_B\Big(\frac{\beta_A e_{A,2t-1}}{1+\beta_A} + B_{2t-1}\Big)\Big(\frac{\beta_B e_{B,2t}}{1+\beta} + B_{2t}\Big) \leq \beta_A \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t-1}}{1+\beta_A} \tag{33}$$

$$\beta_A\Big(\frac{\beta_B e_{B,2t}}{1+\beta_B} + A_{2t}\Big)\Big(\frac{\beta_A e_{A,2t+1}}{1+\beta_A} + B_{2t+1}\Big) \leq \beta_B \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t+1}}{1+\beta_A} \tag{34}$$

$$\beta_B\Big(\frac{\beta_A e_{A,2t-1}}{1+\beta_A} + B_{2t-1}\Big)\Big(\frac{\beta_B e_{B,2t}}{1+\beta} + A_{2t}\Big) \leq \beta_A \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t-1}}{1+\beta_A}. \tag{35}$$

These conditions are not too demanding and are satisfied if, for instance, $\beta_A = \beta_B = \beta$,

$$A_{2t}, B_{2t} < \frac{(1-\beta)e_{B,2t}}{1+\beta} \text{ and } A_{2t+1}, B_{2t+1} < \frac{(1-\beta)e_{A,2t+1}}{1+\beta} \ \forall t.$$

Equilibrium. Let us construct an equilibrium $\left(q_t, (c_{i,t}, l_{i,t})_{i\in I}\right)_t$ as follows.

In Appendix 11, we compute the equilibrium allocations.

$$l_{A,2t-1} = L, l_{A,2t} = 0, \quad l_{B,2t-1} = 0, l_{B,2t} = L \tag{36}$$

$$c_{A,2t} = q_{2t}L + A_{2t}L, \quad c_{B,2t} + q_{2t}L = e_{B,2t} \tag{37}$$

$$c_{A,2t+1} + q_{2t+1}L = e_{A,2t+1}, \quad c_{B,2t+1} = q_{2t+1}L + B_{2t+1}L \tag{38}$$

as well the equilibrium prices

$$q_{2t} = \frac{\beta_B}{1+\beta_B}e_{B,2t} \text{ and } q_{2t+1} = \frac{\beta_A}{1+\beta_A}e_{A,2t+1}$$

for any $t \geq 0$.

We find also the discount factors and the land dividends

$$\gamma_{2t} = \frac{\beta_A u'_A(c_{A,2t})}{u'_A(c_{A,2t-1})} \text{ and } \gamma_{2t+1} = \frac{\beta_B u'_B(c_{B,2t+1})}{u'_B(c_{B,2t})} \tag{39}$$

$$d_{2t} = A_{2t} \text{ and } d_{2t+1} = B_{2t+1} \tag{40}$$

for any $t \geq 0$.

According to Proposition 6, land bubbles exist if and only if $\sum\limits_{t=0}^{\infty} \frac{d_t}{q_t} < \infty$, i.e.,

$$\sum_{t=0}^{\infty} \frac{A_{2t}}{e_{B,2t}} + \sum_{t=0}^{\infty} \frac{B_{2t+1}}{e_{A,2t+1}} < \infty$$

Intuition. This condition may be interpreted that land dividends are low with respect to endowments. In other words, the existence of bubbles requires low dividends. Let us explain the intuition. In the odd periods $(2t+1)$, agent B has no endowments. She wants to smooth consumption over time according to her logarithm utility (which satisfies the Inada conditions), but she cannot transfer her wealth from future to this date.[14] By consequence, she accepts to buy land at a higher price: $q_{2t} \geq e_{B,2t}\beta_B/(1+\beta_B)$, independently on agents' productivity. A lower productivity implies lower dividends and a lower fundamental value of land. As long as dividends tend to zero, the land price remains higher than this fundamental value.

We point our some particular cases of our example.

Example 1 (land bubble with endowment growth)**.** *Consider our example. Assume that* $A_t = B_t = A$ *for any* t. *Then land bubbles exist if and only if*

$$\sum_{t=0}^{\infty} \frac{1}{e_{B,2t}} + \sum_{t=0}^{\infty} \frac{1}{e_{A,2t+1}} < \infty$$

Example 1 illustrates Proposition 8. Thus, under a common stationary production function and $f_i = 0$ for any i, land bubbles may appear if endowments tend to infinity. In this example, we see that land bubbles arise if and only if $\sum\limits_{t=1}^{\infty} 1/q_t < \infty$. By the way, this result also illustrates Corollary 4.

Example 2 (land bubble with collapsing land technologies)**.** *Reconsider our example. If* $e_{A,2t+1} = e_{B,2t} = e > 0$ *for any* t, *then land bubbles emerge if and only if* $\sum_{t=0}^{\infty}(A_{2t}+B_{2t+1}) < \infty$.

[14] Because she is prevented from borrowing.

This result is also related to Bosi et al. (2015) where they show that bubbles in aggregate good arise if the sum of capital returns is finite.

Some interesting remarks deserve mention.

1. **Bubble vs *i*-bubble**. Since $\lim_{t\to\infty}\beta_i^t u_i'(c_{i,t})q_t = 0$ for $i = A, B$, there are no i-bubbles. However, land bubbles may occur. In this case, any individual value of land is identical and equal to the equilibrium price but it may exceed the fundamental value of land.

2. ***i*-bubble and borrowing constraints**. In the above example, borrowing constraints of both agents are binding at infinitely many dates while every individual bubbles are ruled out and land bubbles may or may not appear. This shows that the values of (individual) bubbles are not the shadow prices of binding borrowing constraints.

3. **Pure bubble (or fiat money)**. We consider a particular case: If $A_t = B_t = 0$ for every t. In this case, the fundamental value of land is zero and an equilibrium is bubbly if the prices of land are strictly positive in any period ($q_t > 0$ for any t). This bubble is called *pure bubble* by (Tirole, 1985). Our example shows that equilibria with pure bubble may exist in infinite-horizon general equilibrium models.

   In this case, the land in our model can be interpreted as *fiat money* in Bewley (1980), Santos and Woodford (1997), Pascoa et al. (2011) where they provide some examples where the fiat money price is strictly positive. Our contribution concerns the existence of bubble of assets with positive and endogenous dividends.

4. **Land bubbles vs monotonicity of prices**. Corollary 4 points out that, under stationary technologies, the existence of land bubble entails the divergence of land prices to infinity. However, in our example with non-stationary technologies, the land prices are given by

   $$q_{2t} = \frac{\beta_B}{1+\beta_B}e_{B,2t} \text{ and } q_{2t+1} = \frac{\beta_A}{1+\beta_A}e_{A,2t+1}$$

   and we see that land prices may either increase or decrease or fluctuate over time whenever bubbles exist. Our result generalizes that of Weil (1990) where he gives an example of bubble with decreasing asset prices. His model is a particular case of ours when land gives no longer fruits from some date on: there exists T such that $A_t = B_t = 0$ for any $t \geq T$.

5. **Do the most productive agents produce?** In the above examples, although agents have linear production functions, these functions are different.

   There is a case where the productivity of agent A is higher than that of agent B, i.e., $A_{2t+1} > B_{2t+1}$, but agent A does not produce at date $2t+1$ while agent B produce at this date. For two reasons: (1) agents are prevented from borrowing, (2) agents' endowments change over time. Although A has a higher productivity at date $2t+1$, she has also a higher endowment at this date, but no endowment at date $2t$. So, she may not need to buy land at date $2t$ to produce and transfer wealth from date $2t$ to date $2t+1$. Instead, she sells land at date $2t$ to buy and consume consumption good at date $2t$. Therefore, agent A may not produce at date $2t+1$ even $A_{2t+1} > B_{2t+1}$.

Using similar methods, we may construct other examples of bubbles with non-linear production functions, for example $F_{i,t}(x) = A_{i,t}ln(1+x)$ where $A_{i,t} \geq 0$.

5.1.2 Example of individual land bubbles

The economy's fundamentals. Consider the example in Section 5.1.1. For the sake of simplicity, we assume that $\beta_A = \beta_B =: \beta$.

We add the third agent: agent D. The utility, the rate of time preference, and the technologies of agent D are: $u_D(c) = ln(c), \quad \beta_D = \beta, \quad F_{D,t}(L) = D_t L$.

The endowments $(e_{D,t})_t$ and productivities (D_t) of agents D are defined by

$$\frac{\beta e_{D,t}}{e_{D,t+1}} = \frac{q_t}{q_{t+1} + D_{t+1}} = \gamma_{t+1}$$

where (γ_t) is determined as in Section 5.1.1. We see that such sequences $(e_{D,t})_t$ and (D_t) exist. Indeed, for example, we choose $D_t = d_t$ where d_t is determined as in (40). Then, we choose $(e_{D,t})_t$ such that $\beta e_{D,t} = \gamma_{t+1} e_{D,t+1}$.

Equilibrium: Prices and allocations of agents A and B are as in Section 5.1.1. The allocations of agent D are $c_{D,t} = e_{D,t}$ and $l_{D,t} = 0$ for any t. By using the same argument in Section 5.1.1, it is easy to verify that this system of prices and allocations constitutes an equilibrium.

We observe that agent D does not trade and $\gamma_{D,t} = \beta e_{D,t-1}/e_{D,t} = \gamma_t$ for any t. By consequence, $\lim_{t\to\infty} Q_{D,t} q_t = \lim_{t\to\infty} Q_t q_t > 0$. There is a D - bubble, i.e. the equilibrium price of land is strictly higher than the individual value of land with respect to agent D.

5.2 Land bubbles with short-sales

We now provide examples of bubbles when short-sales are allowed. These new examples particularly contribute to the novelty of our paper.

The economy's fundamentals. Assume that there are two agents (A and B) with a common utility function $u_A(x) = u_B(x) = ln(x)$ but different non-stationary technologies: $F_{A,t}(X) = A_t X, F_{B,t}(X) = B_t X$ with

$$B_{2t} \geq A_{2t}, \quad A_{2t+1} \geq B_{2t+1}$$

for any t. The supply of land is $L = 1$. Borrowing limits are $f_A = f_B = 1$. For simplicity, we assume that $\beta_A = \beta_B = \beta \in (0, 1)$. Applying Proposition 2 allows us obtain the following result.

Example 3 (land bubbles with short-sales)**.** *Let endowments be given by*[15]

$$\begin{aligned} e_{B,2t-1} = e_{A,2t} = 0 \quad \forall t \geq 1 \\ \frac{e_{B,2t} e_{A,2t+1}}{(1+\beta)^2} > \Big(\frac{\beta}{1+\beta} e_{B,2t} + B_{2t}\Big)\Big(\frac{\beta}{1+\beta} e_{A,2t+1} + A_{2t+1}\Big) \quad \forall t \geq 0 \\ \frac{e_{B,2t} e_{A,2t-1}}{(1+\beta)^2} > \Big(\frac{\beta}{1+\beta} e_{B,2t} + B_{2t}\Big)\Big(\frac{\beta}{1+\beta} e_{A,2t-1} + A_{2t-1}\Big) \quad \forall t \geq 1 \end{aligned}$$

Equilibrium prices are determined as follows:

$$p_t = 1, \quad q_0 = \beta(e_{B,0} + B_0), \quad q_{2t} = \frac{\beta}{1+\beta} e_{B,2t}, \quad q_{2t-1} = \frac{\beta}{1+\beta} e_{A,2t-1},$$
$$R_{2t} = \frac{q_{2t} + B_{2t}}{q_{2t-1}}, \quad R_{2t-1} = \frac{q_{2t-1} + A_{2t-1}}{q_{2t-2}}.$$

[15] These conditions are similar to (32, 33, 34, 35) in our example of bubble in models without short-sales.

Allocations are determined by

$$(l_{A,2t-1}, l_{B,2t-1}) = (0,1), \quad (l_{A,2t}, l_{B,2t}) = (1,0) \tag{41}$$

$$a_{A,2t-1} = \frac{q_{2t} + B_{2t}}{R_{2t}} = -a_{B,2t-1}, \quad a_{B,2t} = \frac{q_{2t+1} + A_{2t+1}}{R_{2t+1}} = -a_{A,2t} \quad \forall t \geq 1. \tag{42}$$

Dividends are calculated by

$$d_{2t} = B_{2t} \text{ and } d_{2t+1} = A_{2t+1}. \tag{43}$$

According to Proposition 6, land bubbles exist if and only if

$$\sum_{t=1}^{\infty} \frac{B_{2t}}{e_{B,2t}} + \sum_{t=0}^{\infty} \frac{A_{2t+1}}{e_{A,2t+1}} < \infty. \tag{44}$$

As in economies without-short sales, bubbles may occur if endowments growth without bound and/or TFP tends to zero.

The intuition *of our example: Look at the economy at date $2t$. Agent B knows that she will not have endowment at date $2t+1$: $e_{B,2t+1} = 0$, and hence she wants to transfer her wealth from date $2t$ to date $2t+1$ (she saves at date $2t$). Therefore, she may accept to buy land with a high price or buy financial asset with low interest rates. The same argument applies for the agent A at date $2t+1$. Therefore, the price of land may be higher than its fundamental value or equivalently the bubble component $\lim\limits_{t\to\infty} \frac{q_t}{R_1 R_2 \ldots R_t}$ may be strictly positive.*

Some observations should be mentioned.

1. **Dividends are endogenous**. Comparing our example in this section and that in Section 5.1.1, the technologies of two agents A and B do not change but land dividends change (see (40) and (43)). This difference is from the fact that land dividends are endogenous defined.

2. **With vs without short-sales**. In Examples without short-sale in Section 5.1.1, agents transfer there wealth from one date to the next date by the unique way: buying land. However, in Example 3, they do so by investing in the financial market or buying land. Thanks to the financial market, land is used by the most productive agent in Example 3. This is not true when agents are prevented from borrowing as showed in Section 5.1.1.

6 Extension: a stochastic model

In this section, we will extend our analysis to the stochastic case and discuss the land valuation.

6.1 Framework and basic properties

We follow the literature of infinite-horizon incomplete markets as Magill and Quinzii (1994), Magill and Quinzii (1996), Kubler and Schmedders (2003), Magill and Quinzii (2008) and references therein, or more recently Araujo et al. (2011), Pascoa et al. (2011).

Consider an infinite-horizon discrete time economy where the set of dates is $0, 1, \ldots$ and there is no uncertainty at initial date ($t = 0$). Given a history of realizations of the states of nature for the first $t-1$ dates, with $t \geq 1$, $\bar{s}_t = (s_0, \ldots, s_{t-1})$, there is a finite set $\mathcal{S}(\bar{s}_t)$ of states that may occur at date t. A vector $\xi = (t, \bar{s}_t, s)$, where $t \geq 1$ and $s \in \mathcal{S}(\bar{s}_t)$, is called a *node*. The only node at $t = 0$ is denoted by ξ_0. Let $\mathcal{D}$ be the (countable) event-tree, i.e., the set of all nodes. We denote by $t(\xi)$ the date associated with a node ξ.

Given $\xi := (t, \bar{s}_t, s)$ and $\mu := (t', \bar{s}_{t'}, s')$, we say that μ is a successor of ξ, and we write $\mu > \xi$, if $t' > t$ and the first $t+1$ coordinates of $\bar{s}_{t'}$ are $(\bar{s}_t, s)$. We write $\mu \geq \xi$ to say that either $\mu > \xi$ or $\mu = \xi$.

For each T and ξ, we denote $D(\xi) := \{\mu : \mu \geq \xi\}$ the sub-tree with root ξ; $D_T := \{\xi : t(\xi) = T\}$ the family of nodes with date T; $D^T(\xi) := \bigcup_{t=t(\xi)}^{T} D_t(\xi)$, where $D_T(\xi) := D_T \cap D(\xi)$; $D^T := D^T(\xi_0)$; $\xi^+ := \{\mu \geq \xi : t(\mu) = t(\xi) + 1\}$ the set of immediate successors of ξ; ξ^- the unique predecessor of ξ.

There is a single consumption good at each node. The number m of agents is finite. I denotes the set of agents. At each node ξ, each agent i is endowed $e_{i,\xi} > 0$ units of consumption good.

Each household i takes the sequence of prices $(p, q, R) := (p_\xi, q_\xi, R_\xi)_{\xi \in \mathcal{D}}$ as given and chooses sequences of consumption, land, and asset volume $(c_i, l_i, a_i) := (c_{i,\xi}, l_{i,\xi}, a_{i,\xi})_{\xi \in \mathcal{D}}$ in order to maximizes her intertemporal utility

$$P_i(p, q, R) : \quad \max_{(c_i, l_i, a_i)} \Big[U_i(c_i) := \sum_{\xi \in \mathcal{D}} u_{i,\xi}(c_{i,\xi}) \Big]$$

subject to, for each $\xi \geq \xi_0$,

$$l_{i,\xi} \geq 0 \tag{45}$$

$$p_\xi c_{i,\xi} + q_\xi l_{i,\xi} + p_\xi a_{i,\xi} \leq p_\xi e_{i,\xi} + q_\xi l_{i,\xi^-} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + R_\xi a_{i,\xi^-} \tag{46}$$

$$R_{\xi'} a_{i,\xi} \geq -f_i \big[q_{\xi'} l_{i,\xi} + p_{\xi'} F_{i,\xi'}(l_{i,\xi}) \big] \quad \forall \xi' \in \xi^+, \tag{47}$$

where $l_{i,\xi_0^-} > 0$ is given and $a_{i,\xi_0^-} = 0$. Notice that we allow for non-stationary production functions.

The deterministic model corresponds to the case where $\mathcal{D} = \{0, 1, 2, \ldots\}$ and $u_{i,\xi}(c) = \beta_i^{t(\xi)} u_i(c)$.

Another particular case of our model, where $F_{i,\xi} = 0$, $f_i = 0$ for any i, ξ, and there is no short-sale, corresponds to Pascoa et al. (2011). In this case, land can be interpreted as fiat money. However, Pascoa et al. (2011) assume that agents have money endowments at each node while we consider that agents have land endowments only at initial node.

Since constraint (47) can be interpreted as a collateral constraint, our stochastic model is also related to Gottardi and Kubler (2015) where they construct a tractable model with collateral constraints and complete markets, and provide sufficient conditions for the existence of Markov equilibria. However, when financial markets are incomplete like in our model, as mentioned by Gottardi and Kubler (2015), it is not easy to find out robust equilibrium properties.

If we consider $f_i = 1$, constraint (47) becomes solvency constraint (4) in Chien and Lustig (2010) where they consider a model with a continuum of identical agents and a complete menu of contingent claims. In our model, there are a finite number of heterogeneous agents but financial markets are incomplete.

The economy is denoted by $\mathcal{E}$ characterized by a list of fundamentals

$$\mathcal{E}_s := \Big((u_{i,\xi}, e_{i,\xi}, F_{i,\xi})_{\xi \in \mathcal{D}}, f_i, l_{i,\xi_0^-}, a_{i,\xi_0^-} \Big)_{i \in I}.$$

Definition 6. *Given the economy $\mathcal{E}$. A list $\Big(\bar{p}_\xi, \bar{q}_\xi, \bar{R}_\xi, (\bar{c}_{i,\xi}, \bar{l}_{i,\xi}, \bar{a}_{i,\xi})_{i=1}^m\Big)_{\xi\in\mathcal{D}}$ is an equilibrium if the following conditions are satisfied:*

(i) Price positivity: $\bar{p}_\xi, \bar{q}_\xi, \bar{R}_\xi > 0$ for any ξ.

(ii) Market clearing: at each ξ,

$$\text{good: } \sum_{i=1}^m \bar{c}_{i,\xi} = \sum_{i=1}^m (e_{i,\xi} + F_{i,\xi}(\bar{l}_{i,\xi^-})) \tag{48}$$

$$\text{land: } \sum_{i=1}^m \bar{l}_{i,\xi} = L \tag{49}$$

$$\text{financial asset: } \sum_{i=1}^m \bar{a}_{i,\xi} = 0. \tag{50}$$

(iii) Agents' optimality: for each i, $(\bar{c}_{i,\xi}, \bar{l}_{i,\xi}, \bar{a}_{i,\xi})_{\xi\in\mathcal{D}}$ is a solution of the problem $P_i(\bar{p}, \bar{q}, \bar{R})$.

Some standard assumptions are required in order to get the equilibrium existence.

Assumption 6 (production functions)**.** *For each i and ξ, the function $F_{i,\xi}$ is concave, continuously differentiable, $F'_{i,\xi} > 0$, $F_{i,\xi}(0) = 0$.*

Assumption 7 (endowments)**.** *$l_{i,\xi_0^-} > 0$ and $a_{i,\xi_0^-} = 0$ for any i. $e_{i,t} > 0$ for any i and for any t.*

Assumption 8 (borrowing limits)**.** *$f_i \in (0, 1]$ for any i.*

Assumption 9 (utility functions)**.** *For each i and for each $\xi \in \mathcal{D}$, the function $u_{i,\xi} : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable, concave, $u_{i,\xi}(0) = 0$, $u_{i,\xi}(+\infty) = +\infty$, $u'_{i,\xi} > 0$, $u'_{i,\xi}(0) = \infty$.*

Assumption 10 (finite utility)**.** *For each i,*

$$\sum_{\xi\in\mathcal{D}} u_{i,\xi}(W_\xi) < \infty, \qquad \text{where } W_\xi := \sum_{i=1}^m \Big(e_{i,\xi} + F_{i,\xi}(L)\Big). \tag{51}$$

Proposition 11. *Under assumptions (6) to (10), there exists an equilibrium.*

A proof of Proposition 11 is presented in the online appendix.

Price normalization. Since $p_\xi > 0$ for any ξ, we will normalize by setting $p_\xi = 1$. In this case $(q, R, (c_i, l_i, a_i)_i)$ denotes an equilibrium, where $q_\xi, R_\xi \in (0, \infty)$ are price and return in terms of consumption good.

Proposition 12 (First-order and transversality conditions)**.**

1. *Let $(q, R, (c_i, l_i, a_i)_i)$ be an equilibrium. There exists positive sequences of multipliers $(\lambda_{i,\xi}, \eta_{i,\xi})_{\xi\in\mathcal{D}}$, $(\mu_{i,\xi})_{\xi\in\mathcal{D}:t(\xi)\geq 1}$ such that*

$$\text{FOCs: } u'_{i,\xi}(c_{i,\xi}) = \lambda_{i,\xi} \tag{52}$$

$$\lambda_{i,\xi} = \sum_{\xi'\in\xi^+} (\lambda_{i,\xi'} + \mu_{i,\xi'})R_{\xi'} \tag{53}$$

$$\lambda_{i,\xi} q_\xi = \sum_{\xi'\in\xi^+} (\lambda_{i,\xi'} + f_i\mu_{i,\xi'})(q_{\xi'} + F'_{i,\xi}(l_{i,\xi})) + \eta_{i,\xi} \tag{54}$$

$$\eta_{i,\xi} l_{i,\xi} = 0 \tag{55}$$

$$\mu_{i,\xi'}\Big(R_{\xi'}a_{i,\xi} + f_i\big[q_{\xi'}l_{i,\xi} + F_{i,\xi'}(l_{i,\xi})\big]\Big) = 0 \ \forall \xi' \in \xi^+ \tag{56}$$

$$\text{Transversality condition: } \lim_{T\to\infty} \sum_{\xi\in D_T} \lambda_{i,\xi}(q_\xi l_{i,\xi} + a_{i,\xi}) = 0 \tag{57}$$

2. *If the sequences $(q, R, (c_i, l_i, a_i)_i)$ and $(\lambda_i, \eta_i, \mu_i)_i$ satisfy*
   (a) *$c_{i,\xi}, l_{i,\xi}, \lambda_{i,\xi}, \eta_{i,\xi}, \mu_{i,\xi'} \geq 0$; $q_\xi, R_\xi > 0$ for any $\xi \in \mathcal{D}$ and $\xi' \in \xi^+$.*
   (b) *conditions (46) is binding, and (45), (47), (48), (49), (50) hold;*
   (c) *conditions (52), (53), (54), (55), (56), (57) hold;*
   (d) *$\sum_{\xi\in\mathcal{D}} u_{i,\xi}(c_{i,\xi}) < \infty$ for any i;*

   then $(q, R, (c_i, l_i, a_i)_i)$ is an intertemporal equilibrium.

A proof of Proposition 12 is presented in Appendix 12 for. Notice that our result also holds if production functions are not stationary.

As in the deterministic case, we have $q_\xi \geq f_i \min_{\xi'\in\xi^+} \frac{q_{\xi'} + F'_{i,\xi'}(l_{i,\xi})}{R_{\xi'}}$.[16]

6.2 Land valuation

6.2.1 Individual valuation and bubble

We give an extension of analysis in Section 4.2. For each $\xi' \in \xi^+$, let us denote $P_{\xi\xi'}$ the probability that the successor of ξ is ξ'. We have $\sum_{\xi'\in\xi^+} P_{\xi\xi'} = 1$. According to (54), we have

$$\begin{aligned} q_\xi &= \sum_{\xi'\in\xi^+} \frac{\lambda_{i,\xi'} + f_i\mu_{i,\xi'}}{\lambda_{i,\xi}}(q_{\xi'} + F'_{i,\xi'}(l_{i,\xi})) + \frac{\eta_{i,\xi}}{\lambda_{i,\xi}} \\ &= \sum_{\xi'\in\xi^+} \frac{\lambda_{i,\xi'}}{\lambda_{i,\xi}}\Big(q_{\xi'} + \underbrace{F'_{i,\xi'}(l_{i,\xi}) + \frac{P_{\xi\xi'}\eta_{i,\xi}}{\lambda_{i,\xi'}}}_{\text{Production return}} + \underbrace{\frac{f_i\mu_{i,\xi'}}{\lambda_{i,\xi'}}(q_{\xi'} + F'_{i,\xi'}(l_{i,\xi}))}_{\text{Collateral return}}\Big) \end{aligned} \tag{59}$$

[16]Indeed, since $f_i \leq 1$, we have

$$q_\xi \geq f_i \frac{\sum_{\xi'\in\xi^+} (\lambda_{i,\xi'} + \mu_{i,\xi'})(q_{\xi'} + F'_{i,\xi'}(l_{i,\xi})}{\sum_{\xi'\in\xi^+} (\lambda_{i,\xi'} + \mu_{i,\xi'})R_{\xi'}} \geq f_i \min_{\xi'\in\xi^+} \frac{q_{\xi'} + F'_{i,\xi'}(l_{i,\xi})}{R_{\xi'}}. \tag{58}$$

Denote $d_{i,\xi'} := F'_{i,\xi'}(l_{i,\xi}) + \frac{p_{\xi\xi'}\eta_{i,\xi}}{\lambda_{i,\xi'}} + \frac{f_i\mu_{i,\xi'}}{\lambda_{i,\xi'}}(q_{\xi'} + F'_{i,\xi'}(l_{i,\xi}))$ and $\gamma_{i,\xi'} := \frac{\lambda_{i,\xi'}}{\lambda_{i,\xi}}$, for any $\xi' \in \xi^+$. For each ξ, we denote $Q_{i,\xi} := \prod_{\mu \leq \xi} \gamma_{i,\mu} = \frac{\lambda_{i,\xi}}{\lambda_{i,\xi_0}}$. We have land price decomposition

$$\begin{aligned} q_{\xi_0} &= \sum_{\xi \in \xi_0^+} \gamma_{i,\xi}(q_\xi + d_{i,\xi}) = \sum_{\xi \in \xi_0^+} Q_{i,\xi} d_{i,\xi} + \sum_{\xi \in \xi_0^+} \gamma_{i,\xi} q_\xi \\ &= \sum_{t=1}^{T} \sum_{\xi \in D_t} Q_{i,\xi} d_{i,\xi} + \sum_{\xi \in D_T} Q_{i,\xi} q_\xi \end{aligned} \tag{60}$$

Definition 7 (individual bubble). *$FV_i := \sum_{t=1}^{\infty} \sum_{\xi \in D_t} Q_{i,\xi} d_{i,\xi}$ is the i-fundamental value of land. We say that a i-land bubble exists if $q_{\xi_0} > FV_i$.*

This can be viewed as a generalized version of fiat money valuation in Pascoa et al. (2011) which corresponds to the case where $F_{i,\xi} = 0$ and short-sales are not allowed i.e., $f_i = 0$.

The following result shows the role of heterogeneity of agents.

Proposition 13. *1. If there is $M > 0$ such that $\lambda_{i,\xi}/\lambda_{j,\xi} < M$ for any i, j and for any ξ, then $q_0 = FV_i$ for any i.*

2. If there exists t_0 such that $\gamma_{i,\xi} = \gamma_{j,\xi}$ for any i, j, and for any ξ, then $q_0 = FV_i$ for any i.

Proof. Le us prove point 1. Given i. We have $\lambda_{i,\xi}(q_\xi l_{j,\xi} + a_{j,\xi}) \leq M\lambda_{j,\xi}(q_\xi l_{j,\xi} + a_{j,\xi})$, so

$$\lim_{T \to \infty} \sum_{\xi \in D_T} \lambda_{i,\xi}(q_\xi l_{j,\xi} + a_{j,\xi}) = 0 \tag{61}$$

for any j. Taking the sum over j, we have $\lim_{T\to\infty} \sum_{\xi \in D_T} \lambda_{i,\xi} q_\xi L = 0$ which implies that $q_0 = FV_i$.

Point 2 is a direct consequence of point 1. □

6.2.2 In search of a theory of valuation

Given a node ξ, since $\sum_i l_{i,\xi} = L > 0$, there exists $i = i(\xi)$ such that $l_{i,\xi} > 0$. By combining this with condition (54) in Proposition 12, we have

$$q_\xi = \sum_{\xi' \in \xi^+} \frac{\lambda_{i,\xi'} + f_i \mu_{i,\xi'}}{\lambda_{i,\xi}} \Big(q_{\xi'} + F'_{i,\xi'}(l_{i,\xi}) \Big). \tag{62}$$

Corollary 7. *Consider a particular case (the Lucas tree): $F_{i,\xi}(x) = d_\xi x$ for any i and for any ξ. For each ξ, there is a state-price process $(\gamma_{\xi'})_{\xi' \in \xi^+}$ such that*

$$q_\xi = \sum_{\xi' \in \xi^+} \gamma_{\xi'} \Big(q_{\xi'} + d_{\xi'} \Big) \tag{63}$$

Equation (63) is the traditional intertemporal no-arbitrage. According to this result, we can apply the approach of Santos and Woodford (1997). In the deterministic case, we have proved that there is a unique state-price process (γ_t) given by $\gamma_t = 1/R_t$ for any $t \geq 0$, where R_t is the interest rate at date t. However, in the stochastic case with incomplete markets, the uniqueness of (γ_ξ) is not ensured because, for each i, $\big(\frac{\lambda_{i,\xi'} + f_i\mu_{i,\xi'}}{\lambda_{i,\xi}}\big)_{\xi' \in \xi^+}$ is a state-prices process.

In the standard case where assets have exogenous dividends, it is sufficient to define state-price process in order to evaluate assets as in Santos and Woodford (1997), Montrucchio (2004). In our model, not only state-prices process but also dividends are to defined. So, what is the dividend of land? It is value added that land brings for the economy. Since land can be used by any agent, dividend of land at note $\xi' \in \xi^+$ should be greater than $\min_i F'_{i,\xi'}(l_{i,\xi})$. This idea leads to the following concept.

Definition 8. *Consider an equilibrium. Fixe a node ξ. $\Gamma_\xi := (\gamma_{\xi'}, d_{\xi'})_{\xi' \in \xi}$ is called a state-price (or discount factor) and land dividend process if*

$$d_{\xi'} \geq \min_i F'_{i,\xi'}(l_{i,\xi}) \text{ for any } \xi' \in \xi^+ \tag{64}$$

$$q_\xi = \sum_{\xi' \in \xi^+} \gamma_{\xi'} \Big(q_{\xi'} + d_{\xi'} \Big) \tag{65}$$

According to (59) or (62), the set of all state-price and land dividend process is not empty. Definition 8 covers the traditional intertemporal pricing of assets with exogenous dividends (Santos and Woodford, 1997; Montrucchio, 2004). It also covers the concepts of (individual) dividends of land in our deterministic case and in Section 6.2.1 for the stochastic case. Here, we propose an approach for valuation of productive assets which are more general than the fiat money in Pascoa et al. (2011).

Miao and Wang (2012, 2015) also consider valuation of stocks with endogenous dividends. However, their approach cannot be applied for valuation of land in our model because land can be used by many agents while a stock in Miao and Wang (2012, 2015) is issued by only one firm and stock dividends are taken as given by other agents.

Given a process of state-price and land dividend $\Gamma := (\gamma_\xi, d_\xi)_{\xi \in \mathcal{D}}$, let us denote $Q_\xi := \prod_{\xi' \leq \xi} \gamma_{\xi'}$. Hence, we can provide

$$q_{\xi_0} = \sum_{t=1}^{T} \sum_{\xi \in D_t} Q_\xi d_\xi + \sum_{\xi \in D_T} Q_\xi q_\xi \tag{66}$$

Definition 9. *Given a process of state-price and land dividend $\Gamma := (\gamma_\xi, d_\xi)_{\xi \in \mathcal{D}}$. The fundamental value of land associated to this process is defined by $FV_\Gamma := \sum_{t=1}^{\infty} \sum_{\xi \in D_t} Q_\xi d_\xi$. We say that a Γ-land bubble exists if $q_{\xi_0} > FV_\Gamma$.*

One may ask whether we can choose $d_\xi = 0$ for any ξ, which implies the fundamental value of land equals zero, and then say that bubbles always exist. This cannot be done because condition (64) in Definition 8 must be respected.

Before providing results on land-price bubbles, we present the assumption of uniform impatience mentioned in Levine and Zame (1996), Magill and Quinzii (1994), Santos and Woodford (1997), Pascoa et al. (2011).

Given a consumption plan $c = (c_\mu)_{\mu \in \mathcal{D}}$, a node ξ, a vector $(\gamma, \delta) \in (0,1) \times \mathbb{R}_+$, we define another consumption plan, called $z = z(c, \xi, \gamma, \delta)$, by

$$z_\mu = c_\mu \ \forall \mu \in \mathcal{D} \setminus D(\xi) \tag{67}$$

$$z_\xi = c_\xi + \delta \tag{68}$$

$$z_\mu = \gamma c_\mu \ \forall \mu > \xi. \tag{69}$$

Assumption 11 (Uniform impatience)**.** *There are $\gamma \in (0,1)$ such that for all consumption plan $c = (c_\mu)$ with $0 \leq c_\mu \leq W_\mu \ \forall \mu \in \mathcal{D}$, we have*

$$U_i\Big(z(c, \xi, \gamma', W_\xi)\Big) > U_i(c) \ \forall i, \forall \xi \in \mathcal{D}, \forall \gamma' \in [\gamma, 1). \tag{70}$$

One can prove, by using the same argument in Proposition 1 in Pascoa et al. (2011), that when $(W_\xi)_{\xi\in\mathcal{D}}$ is bounded, the standard utility function $U_i(c) = \sum_\xi \beta_i^{t(\xi)} P_{i,\xi} u_i(c_\xi)$, where $\beta_i \in (0,1)$ and $P_{i,\xi}$ (the probability to reach node ξ) is strictly positive, satisfies Assumption 11.

The main contribution of this section is to provide conditions under which bubbles are ruled out.

Proposition 14. *Let Assumption 11 be satisfied. Given an equilibrium* $(q, R, (c_i, l_i, a_i)_i)$.

1. *(Endogenous condition.) For any* Γ *such that* $\lim_{t\to\infty} \sum_{\xi\in D_t} Q_\xi W_\xi = 0$*, we have* $q_{\xi_0} = FV_\Gamma$ *.*

2. *(Exogenous condition.) Assume that* $\sup_\xi W_\xi < \infty$*. Assume also that* $\inf_{i,\xi} F'_{i,\xi}(L) > 0$*. For any process of state-price and land dividend, there is no land asset bubble, i.e.,* $q_{\xi_0} = FV_\Gamma \ \forall \Gamma$.

Let us explain the intuition of point 1 of Proposition Proposition 14. Thanks to uniform impatience, the ratio of savings to wealth is uniformly bounded. When the discounted value of aggregate wealth is vanishing at infinity, the discounted value of aggregate land quantity at infinity must be zero, i.e., bubbles are ruled out.

Some comments should be mentioned.

- Point 1 of Proposition 14 is a generalized version of Theorem 3.3 in Santos and Woodford (1997). The uniform impatience is required to obtain these results. However, there are two main differences: (1) we deal with endogenous dividends while Santos and Woodford (1997) works with exogenous dividends, and (ii) we only need $\lim_{t\to\infty} \sum_{\xi\in D_t} Q_\xi W_\xi = 0$ while Santos and Woodford (1997) requires $\sum_{t\geq 1} \sum_{\xi\in D_t} Q_\xi W_\xi < \infty$.

- Point 2 of Proposition 14 complements Proposition 8 in the current paper. Proposition 14 needs the uniform impatience but borrowing limits (f_i) can be arbitrary in $[0,1]$ while Proposition 8 do not need the uniform impatience but there is no uncertainty and the financial system must be good enough in the sense that $f_i = 1$ for any i.

- Corollary 1 in Pascoa et al. (2011) indicates that there exists a plan of non-arbitrage deflators for which fiat money has a bubble. However, under conditions of point 2 of Proposition 14, there is no bubble of land for any process of state-prices and dividends. So, the insight in Corollary 1 in Pascoa et al. (2011) may not hold when we work with productive assets such as land in our model.

7 Conclusion

We have built dynamic general equilibrium models with heterogeneous agents and incomplete financial markets, which cover a large class of models used in macroeconomics. First-order and transversality conditions have been proved. Contrary to standard capital accumulation models à la Ramsey, in our model the most patient may not hold the entire stock of land in the long run.

Our paper has provided an approach to the valuation of land. In a bounded economy with stationary production functions and uniform impatience, for any process of state-prices and dividends the price of land equals its fundamental value associated to this process; this holds whatever the level of borrowing limit and of market incompleteness. A number of examples of (individual) bubbles are provided in economies with and without short-sales. Our approach can be used to evaluate other kinds of asset or input such as house or physical capital.

8 Appendix: Proofs for Section 2.2

Proof of part 1 of Proposition 2. Let us prove the transversality condition.[17] Notice that our method here is different from Araujo et al. (2011). Denote $x_i := (l_i, a_i) = (l_{i,t}, a_{i,t})_t$. We say that x_i is feasible if, for every t, we have $l_{i,t} \geq 0$ and

$$R_{t+1}a_{i,t} \geq -f_i(q_{t+1}l_{i,t} + F_i(l_{i,t}))$$
$$q_t l_{i,t} + a_{i,t} \leq e_{i,t} + q_t l_{i,t-1} + F_i(l_{i,t-1}) + R_t a_{i,t-1}.$$

We claim that: if x_i is feasible, then $(x_{i,0}, \ldots, x_{i,t-1}, \lambda x_{i,t}, \lambda x_{i,t+1}, \ldots)$ is also feasible for each $t \geq 1$ and $\lambda \in [0,1]$.

We have to prove that:

$$q_t \lambda l_{i,t} + \lambda a_{i,t} \leq e_{i,t} + q_t l_{i,t-1} + F_i(l_{i,t-1}) + R_t a_{i,t-1} \quad (71)$$

and

$$\lambda q_{s+1} l_{i,s+1} + \lambda a_{i,s+1} \leq e_{i,s+1} + \lambda q_{s+1} l_{i,s} + F_i(\lambda l_{i,s}) + \lambda R_{s+1} a_{i,s} \quad (72)$$
$$\lambda R_{s+1} a_{i,s} + f_i(\lambda q_{s+1} l_{i,s} + F_i(\lambda l_{i,s})) \geq 0 \quad (73)$$

for each $s \geq t$.

(72) and (73) are proved by using the fact that $F_i(\lambda x) \geq \lambda F_i(x)$ for every $\lambda \in [0,1]$.

(71) is satisfied if $q_t l_{i,t} + a_{i,t} < 0$. If $q_t l_{i,t} + a_{i,t} \geq 0$, we have

$$q_t \lambda l_{i,t} + \lambda a_{i,t} \leq q_t l_{i,t} + a_{i,t} \leq e_{i,t} + q_t l_{i,t-1} + F_i(l_{i,t-1}) + R_t a_{i,t-1}.$$

So, our claim is proved.

By using the same argument in Theorem 2.1 in Kamihigashi (2002),[18] we obtain that $\limsup_{t\to\infty} \lambda_{i,t}(q_t l_{i,t} + a_{i,t}) \leq 0$.

According to FOCs, we now have

$$\lambda_{i,t}(c_{i,t} + q_t l_{i,t} + a_{i,t}) = \lambda_{i,t}(e_{i,t} + q_t l_{i,t-1} + F_i(l_{i,t-1}) + R_t a_{i,t-1}) \quad (74)$$
$$\lambda_{i,t} a_{i,t} = (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} a_{i,t} \quad (75)$$
$$\lambda_{i,t} q_t l_{i,t} = (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + F_i'(l_{i,t})) l_{i,t} \quad (76)$$
$$\mu_{i,t+1}\Big(R_{t+1} a_{i,t} + f_i(q_{t+1} l_{i,t} + F_i(l_{i,t}))\Big) = 0. \quad (77)$$

(75) and (76) imply that

$$\begin{aligned}
&\lambda_{i,t}(q_t l_{i,t} + a_{i,t}) \\
&= (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + F_i'(l_{i,t})) l_{i,t} + (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} a_{i,t} \\
&= \lambda_{i,t+1} R_{t+1} a_{i,t} + \lambda_{i,t+1}\big(q_{t+1} + F_i'(l_{i,t})\big) l_{i,t} \\
&+ \mu_{i,t+1}\Big(R_{t+1} a_{i,t} + f_i\big(q_{t+1} + F_i'(l_{i,t})\big) l_{i,t}\Big).
\end{aligned}$$

[17] FOCs are obtained by applying the proof of Proposition 12 in Appendix 12.

[18] Kamihigashi (2002) only considers positive allocations while $a_{i,t}$ may be negative in our model.

Therefore, by combining this with (77), we get that

$$
\begin{aligned}
&\lambda_{i,t}(q_t l_{i,t} + a_{i,t}) - \lambda_{i,t+1} R_{t+1} a_{i,t} - \lambda_{i,t+1}\big(q_{t+1} l_{i,t} + F_i(l_{i,t})\big) \\
&= \lambda_{i,t+1}\big(q_{t+1} + F_i'(l_{i,t})\big) l_{i,t} + \mu_{i,t+1}\Big(R_{t+1} a_{i,t} + f_i\big(q_{t+1} + F_i'(l_{i,t})\big) l_{i,t}\Big) \\
&\quad - \lambda_{i,t+1}\big(q_{t+1} l_{i,t} + F_i(l_{i,t})\big) && (78) \\
&= -\lambda_{i,t+1}(F_i(l_{i,t}) - l_{i,t} F_i'(l_{i,t})) + \mu_{i,t+1}\Big(R_{t+1} a_{i,t} + f_i\big(q_{t+1} + F_i'(l_{i,t})\big) l_{i,t}\Big) \\
&- \mu_{i,t+1}\Big(R_{t+1} a_{i,t} + f_i\big(q_{t+1} l_{i,t} + F_i(l_{i,t})\big)\Big) && (79) \\
&= -\lambda_{i,t+1}(F_i(l_{i,t}) - l_{i,t} F_i'(l_{i,t})) - f_i \mu_{i,t+1}(F_i(l_{i,t}) - l_{i,t} F_i'(l_{i,t})) && (80)
\end{aligned}
$$

By summing (74) from $t = 0$ to T, and then using (78), we obtain that

$$
\begin{aligned}
\sum_{t=0}^{T} \lambda_{i,t} c_{i,t} + \lambda_{i,T}(q_T l_{i,T} + a_{i,T}) &= \sum_{t=0}^{T} \lambda_{i,t} e_{i,t} + \lambda_{i,0}(q_0 l_{i,-1} + F_i(l_{i,-1}) + R_0 a_{i,-1}) \\
&+ \sum_{t=1}^{T} (\lambda_{i,t} + f_i \mu_{i,t})(F_i(l_{i,t-1}) - l_{i,t-1} F_i'(l_{i,t-1})). \qquad (81)
\end{aligned}
$$

Under Assumption (10), the utility of agent i is finite, thus we have

$$
\sum_{t=0}^{\infty} \lambda_{i,t} c_{i,t} = \sum_{t=0}^{\infty} \beta_i^t u_i'(c_{i,t}) c_{i,t} \leq \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) < \infty. \qquad (82)
$$

Combining this with the fact that $\limsup_{t\to\infty} \lambda_{i,t}(q_t l_{i,t} + a_{i,t}) \leq 0$, and (81), we obtain that there exists the following sum

$$
\sum_{t=0}^{\infty} \lambda_{i,t} e_{i,t} + \sum_{t=1}^{\infty} (\lambda_{i,t} + f_i \mu_{i,t})(F_i(l_{i,t-1}) - l_{i,t-1} F_i'(l_{i,t-1})) < \infty.
$$

We now use (81) to get that $\lim_{t\to\infty} \lambda_{i,t}(q_t l_{i,t} + a_{i,t})$ exists and it is non positive.

We again use (74) and note that $q_t l_{i,t-1} + F_i(l_{i,t-1}) + R_t a_{i,t-1} \geq 0$ (because of borrowing constraint) to obtain that $\liminf_{t\to\infty} \lambda_{i,t}(c_{i,t} + q_t l_{i,t} + a_{i,t}) \geq 0$.

(82) implies that $\lim_{t\to\infty} \lambda_{i,t} c_{i,t} = 0$. As a result, we get $\liminf_{t\to\infty} \lambda_{i,t}(q_t l_{i,t} + a_{i,t}) \geq 0$. Therefore, we have $\lim_{t\to\infty} \lambda_{i,t}(q_t l_{i,t} + a_{i,t}) = 0$ and then

$$
\begin{aligned}
\infty > \sum_{t=0}^{\infty} \lambda_{i,t} c_{i,t} &= \sum_{t=0}^{\infty} \lambda_{i,t} e_{i,t} + \sum_{t=1}^{\infty} (\lambda_{i,t} + f_i \mu_{i,t})(F_i(l_{i,t-1}) - l_{i,t-1} F_i'(l_{i,t-1})) \\
&+ \lambda_{i,0}(q_0 l_{i,-1} + F_i(l_{i,-1}) + R_0 a_{i,-1}). \qquad (83)
\end{aligned}
$$

□

Proof of part 2 of Proposition 2. Before proving this part, we should notice that this result requires neither $u_i(0) = 0$ nor $u_i'(0) = \infty$. Let us prove our result. It is sufficient to prove the optimality of (c_i, a_i, l_i) for each i. Let $(c_i', a_i', l_i') \geq 0$ be a plan satisfying all budget and borrowing constraints and $l_{i,-1}' - l_{i,-1} = 0 = a_{i,-1}' = a_{i,-1}$. We have

$$
\sum_{t=0}^{T} \beta_i^t (u_i(c_{i,t}) - u_i(c_{i,t}')) \geq \sum_{t=0}^{T} \beta_i^t u_i'(c_{i,t})(c_{i,t} - c_{i,t}') = \sum_{t=0}^{T} \lambda_{i,t}(c_{i,t} - c_{i,t}')
$$

According to FOCs, we now have

$$\lambda_{i,t}(c'_{i,t} + q_t l'_{i,t} + a'_{i,t}) = \lambda_{i,t}(e_{i,t} + q_t l'_{i,t-1} + F_i(l'_{i,t-1}) + R_t a'_{i,t-1}) \tag{84}$$

$$\lambda_{i,t} a'_{i,t} = (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} a'_{i,t} \tag{85}$$

$$\lambda_{i,t} q_t l'_{i,t} = (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + F'_i(l_{i,t})) l'_{i,t} + \eta_{i,t} l'_{i,t} \tag{86}$$

$$\mu_{i,t+1}\Big(R_{t+1} a_{i,t} + f_i(q_{t+1} l_{i,t} + F_i(l_{i,t}))\Big) = 0. \tag{87}$$

(85) and (86) imply that

$$\begin{aligned}
&\lambda_{i,t}(q_t l'_{i,t} + a'_{i,t}) \\
&= (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + F'_i(l_{i,t})) l'_{i,t} + \eta_{i,t} l'_{i,t} + (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} a'_{i,t} \\
&= \lambda_{i,t+1} R_{t+1} a'_{i,t} + \lambda_{i,t+1}\big(q_{t+1} + F'_i(l_{i,t})\big) l'_{i,t} + \eta_{i,t} l'_{i,t} \\
&+ \mu_{i,t+1}\Big(R_{t+1} a'_{i,t} + f_i\big(q_{t+1} + F'_i(l_{i,t})\big) l'_{i,t}\Big).
\end{aligned}$$

Therefore, by combining this with (87), we get that

$$\begin{aligned}
&\lambda_{i,t}(q_t l'_{i,t} + a'_{i,t}) - \lambda_{i,t+1} R_{t+1} a'_{i,t} - \lambda_{i,t+1}\big(q_{t+1} l'_{i,t} + F_i(l'_{i,t})\big) \\
&= \lambda_{i,t+1}\big(q_{t+1} + F'_i(l_{i,t})\big) l'_{i,t} + \eta_{i,t} l'_{i,t} + \mu_{i,t+1}\Big(R_{t+1} a'_{i,t} + f_i\big(q_{t+1} + F'_i(l_{i,t})\big) l'_{i,t}\Big) \\
&\quad - \lambda_{i,t+1}\big(q_{t+1} l'_{i,t} + F_i(l'_{i,t})\big) \\
&= -\lambda_{i,t+1}(F_i(l'_{i,t}) - l'_{i,t} F'_i(l_{i,t})) + \eta_{i,t} l'_{i,t} + \mu_{i,t+1}\Big(R_{t+1} a'_{i,t} + f_i\big(q_{t+1} + F'_i(l_{i,t})\big) l'_{i,t}\Big).
\end{aligned} \tag{88}$$

According to budget constraints, we have

$$\begin{aligned}
&\lambda_{i,t+1}(e_{i,t+1} - c'_{i,t+1}) + \lambda_{i,t}(q_t l'_{i,t} + a'_{i,t}) - \lambda_{i,t+1}(q_{t+1} l'_{i,t+1} + a'_{i,t+1}) \\
&= \lambda_{i,t}(q_t l'_{i,t} + a'_{i,t}) - \lambda_{i,t+1} R_{t+1} a'_{i,t} - \lambda_{i,t+1}\big(q_{t+1} l'_{i,t} + F_i(l'_{i,t})\big)
\end{aligned} \tag{89}$$

$$\lambda_{i,0} p_0 (e_{i,0} - c'_{i,0}) - \lambda_{i,0}(q_0 l'_{i,0} + a'_{i,0}) + \lambda_{i,0}\big(q_0 l'_{i,-1} + F_i(l'_{i,-1})\big) = 0. \tag{90}$$

By summing these constraints and using (88), we obtain that

$$\begin{aligned}
&\Big[\sum_{t=0}^{T} \lambda_{i,t}(e_{i,t} - c'_{i,t})\Big] - \lambda_{i,T}(q_T l'_{i,T} + a'_{i,T}) + \lambda_{i,0}\big(q_0 l'_{i,-1} + F_i(l'_{i,-1})\big) \\
&= \sum_{t=1}^{T} \Big[-\lambda_{i,t}\big(F_i(l'_{i,t-1}) - l'_{i,t-1} F'_i(l_{i,t-1})\big) + \eta_{i,t-1} l'_{i,t-1} + \mu_{i,t}\big(R_t a'_{i,t-1} + f_i\big(q_t + F'_i(l_{i,t-1})\big) l'_{i,t-1}\Big]
\end{aligned}$$

Since this is satisfied for any feasible allocation (c'_i, a'_i, l'_i), this also holds for the allocation (c_i, a_i, l_i). Consequently, we get

$$\begin{aligned}
&\Big[\sum_{t=0}^{T} \lambda_{i,t}(c_{i,t} - c'_{i,t})\Big] \\
&= \lambda_{i,T}(q_T l'_{i,T} + a'_{i,T}) - \lambda_{i,T}(q_T l_{i,T} + a_{i,T}) \\
&+ \sum_{t=1}^{T} \Big[-\lambda_{i,t}\big(F_i(l'_{i,t-1}) - l'_{i,t-1} F'_i(l_{i,t-1})\big) + \eta_{i,t-1} l'_{i,t-1} + \mu_{i,t}\big(R_t a'_{i,t-1} + f_i\big(q_t + F'_i(l_{i,t-1})\big) l'_{i,t-1}\Big] \\
&- \sum_{t=1}^{T} \Big[-\lambda_{i,t}\big(F_i(l_{i,t-1}) - l_{i,t-1} F'_i(l_{i,t-1})\big) + \eta_{i,t-1} l_{i,t-1} + \mu_{i,t}\big(R_t a_{i,t-1} + f_i\big(q_t + F'_i(l_{i,t-1})\big) l_{i,t-1}\Big] \\
&\geq -\lambda_{i,T}(q_T l_{i,T} + a_{i,T}) + \sum_{t=1}^{T} \lambda_{i,t}\Big[F_i(l_{i,t-1}) - F_i(l'_{i,t-1}) - (l_{i,t-1} - l'_{i,t-1}) F'(l_{i,t-1})\Big] \\
&- \sum_{t=1}^{T} \mu_{i,t}\big(R_t a_{i,t-1} + f_i\big(q_t + F'_i(l_{i,t-1})\big) l_{i,t-1}
\end{aligned}$$

Since F_i is concave, it is easy to see that

$$F_i(l_{i,t}) - F_i(l'_{i,t}) \geq (l_{i,t} - l'_{i,t})F'(l_{i,t}) \ \forall t$$
$$\mu_{i,t}\big(R_t a_{i,t-1} + f_i\big(q_t + F'_i(l_{i,t-1})\big)l_{i,t-1} \leq \mu_{i,t}\big(R_t a_{i,t-1} + f_i\big(q_t l_{i,t-1} + F_i(l_{i,t-1})\big) = 0.$$

Thus, we obtain

$$\sum_{t=0}^{T} \beta_i^t (u_i(c_{i,t}) - u_i(c'_{i,t})) \geq \sum_{t=0}^{T} \lambda_{i,t}(c_{i,t} - c'_{i,t}) \geq -\lambda_{i,T}(q_T l_{i,T} + a_{i,T})$$

By combining this with (14) and the fact that $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) < \infty$, we conclude the optimality of (c_i, a_i, l_i). □

Proof of Corollary 1. Assume that there exists $\lim_{t\to\infty} \Big(Q_t a_{i,t} + f_i Q_{t+1}\big[q_{t+1}l_{i,t} + F_i(l_{i,t})\big]\Big) > 0$. Hence, there exists a date $T \geq 1$ such that borrowing constraint (4) is not binding for every $t \geq T$. Therefore, $\lambda_{i,t} = \lambda_{i,t+1}R_{t+1}$ for every $t \geq T$. By consequence, there exists a constant $C_i \in (0, \infty)$ such that $Q_t = C_i\lambda_{i,t}$ for every $t \geq T$. According to transversality condition (14), we get $\lim_{t\to\infty} Q_t\big(a_{i,t} + q_t l_{i,t}\big) = 0$.

By combining (15) and the fact that $Q_t = C_i\lambda_{i,t}$ for every $t \geq T$, we obtain $\lim_{t\to\infty} Q_t c_{i,t} = \lim_{t\to\infty} Q_t e_{i,t} = 0$. Therefore, by using budget constraints, we get

$$\lim_{t\to\infty} Q_t\big(R_t a_{i,t-1} + q_t l_{i,t-1} + F_i(l_{i,t-1})\big) = 0.$$

Since $f_i \in [0,1]$ and $Q_t R_t = Q_{t-1}$, we obtain the statement (b).

Condition (17) is proved by using the same argument. □

9 Appendix: Proofs for Section 3

Proof of Lemma 2. According to (11), we obtain $q_t \geq \gamma_{t+1}\big(q_{t+1} + \underline{d}_{t+1}\big)$.

We prove the second inequality. We see that there exists an agent, say i, such that $l_{i,t} > 0$. Thus, $\eta_{i,t} = 0$. Therefore, we have

$$\begin{aligned} \lambda_{i,t}q_t &= (\lambda_{i,t+1} + f_i\mu_{i,t+1})(q_{t+1} + F'_i(l_{i,t})) \\ &\leq (\lambda_{i,t+1} + \mu_{i,t+1})(q_{t+1} + F'_i(l_{i,t})) \leq \frac{\lambda_{i,t}}{R_{t+1}}(q_{t+1} + \bar{d}_{t+1}). \end{aligned} \tag{91}$$

By combining with (16), we get the second inequality in (18).

We now prove (19). According to FOCs, we get

$$q_t = \frac{\lambda_{i,t+1} + f_i\mu_{i,t+1}}{\lambda_{i,t+1} + \mu_{i,t+1}} \frac{\lambda_{i,t+1} + \mu_{i,t+1}}{\lambda_{i,t}}(q_{t+1} + F'_i(l_{i,t})) + \frac{\eta_{i,t}}{\lambda_{i,t}} \tag{92}$$
$$= \frac{\lambda_{i,t+1} + f_i\mu_{i,t+1}}{\lambda_{i,t+1} + \mu_{i,t+1}} \frac{1}{R_{t+1}}(q_{t+1} + F'_i(l_{i,t})) + \frac{\eta_{i,t}}{\lambda_{i,t}} \tag{93}$$
$$\geq f_i \frac{1}{R_{t+1}}(q_{t+1} + F'_i(l_{i,t})). \tag{94}$$

Therefore, we obtain (19). □

Proof of Lemma 3. According to (18), we obtain $d_{t+1} \leq \bar{d}_{t+1}$.

Since $f_i = 1$ for any i or (4) is not binding for any i, we always have $\mu_{i,t+1} = f_i\mu_{i,t+1}$ for every i. So, we get

$$q_t = \gamma_{t+1}\big(q_{t+1} + F'_i(l_{i,t})\big) + \frac{\eta_{i,t}}{\lambda_{i,t}} \geq \gamma_{t+1}\big(q_{t+1} + F'_i(l_{i,t})\big)$$

for any i. Therefore $d_{t+1} \geq \bar{d}_{t+1}$. As a result, we have $d_{t+1} = \bar{d}_{t+1}$. □

Proof of Proposition 3. Since $l_{i,t} > 0$ at equilibrium, we have $\eta_{i,t} = 0$. By consequence, we obtain, for every i, t,

$$\lambda_{i,t} p = (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} \tag{95}$$

$$\lambda_{i,t} q_t = (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + F_i'(l_{i,t})). \tag{96}$$

We see that, for every i, t,

$$\begin{aligned} q_t = \frac{\lambda_{i,t+1} + f_i \mu_{i,t+1}}{\lambda_{i,t}}(q_{t+1} + F_i'(l_{i,t})) &\leq \frac{\lambda_{i,t+1} + \mu_{i,t+1}}{\lambda_{i,t}}(q_{t+1} + F_i'(l_{i,t})) \\ &= \gamma_{t+1}(q_{t+1} + F_i'(l_{i,t})) \end{aligned}$$

Therefore, we obtain that $q_t \leq \gamma_{t+1}\big(q_{t+1} + \underline{d}_{t+1}\big)$. By combining with (18), we have

$$q_t = \gamma_{t+1}\big(q_{t+1} + \underline{d}_{t+1}\big).$$

As a result, we get that $d_{t+1} = \underline{d}_{t+1}$. □

Proof of Proposition 4. According to FOCs, we obtain

$$1 = \frac{\lambda_{i,t+1} + f_i \mu_{i,t+1}}{\lambda_{i,t+1} + \mu_{i,t+1}} \frac{q_{t+1} + F_i'(l_{i,t})}{q_{t+1} + d_{t+1}} + \frac{\eta_{i,t}}{\lambda_{i,t} q_t} \tag{97}$$

If $l_{i,t} > 0$, then $\eta_{i,t} = 0$. By combining (97) with $f_i \leq 1$, we get $d_{t+1} \leq F_i'(l_{i,t})$.

We now assume that $d_{t+1} < F_i'(l_{i,t})$. If (4) is not binding, we have $\mu_{i,t+1} = 0$ which implies that $d_{t+1} \geq F_i'(l_{i,t})$, contradiction. □

Proof of Proposition 5. Since $f_j = 1$, condition (11) implies that

$$\frac{q_t}{q_{t+1} + F_j'(l_{j,t})} \geq \frac{\lambda_{j,t+1} + \mu_{j,t+1}}{\lambda_{j,t}} = \frac{1}{R_{t+1}}.$$

Assume that $l_{i,t} > 0$, we have $\eta_{i,t} = 0$ which implies that

$$\frac{q_t}{q_{t+1} + F_i'(l_{i,t})} = \frac{\lambda_{i,t+1} + f_i \mu_{i,t+1}}{\lambda_{i,t}} \leq \frac{1}{R_{t+1}} \leq \frac{q_t}{q_{t+1} + F_j'(l_{j,t})}.$$

Therefore, $F_j'(L) \leq F_j'(l_{j,t}) \leq F_i'(l_{i,t}) < F_i'(0)$, contradiction. □

Proof of Lemma 4. Let $\big(q, R, (c_i, l_i, a_i), (c_j, l_j, a_j)\big)$ be a steady state equilibrium. According to Proposition 2, we rewrite the system (9, 10, 11, 12, 13)

$$\begin{aligned} &\beta_i^t u_i'(c_{i,t}) = \lambda_{i,t} \\ &1 = R_{t+1}\Big(\frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + \frac{\mu_{i,t+1}}{\lambda_{i,t}}\Big) \\ &q_t = \big(\frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + f_i \frac{\mu_{i,t+1}}{\lambda_{i,t}}\big)\big(q_{t+1} + F_i'(l_{i,t})\big) + \frac{\eta_{i,t+1}}{\lambda_{i,t}} \\ &\eta_{i,t} l_{i,t} = 0 \\ &\mu_{i,t+1}\Big(R_{t+1} a_{i,t} + f_i\big[q_{t+1} l_{i,t} + F_i(l_{i,t})\big]\Big) = 0 \end{aligned}$$

Let us denote $x_{i,t} := \dfrac{\mu_{i,t+1}}{\lambda_{i,t}}$, $\sigma_{i,t} := \dfrac{\eta_{i,t}}{\lambda_{i,t}}$.

At steady state, we have

$$\begin{aligned} 1 &= R(\beta_i + x_i) \\ q &= (\beta_i + f_i x_i)(q + F_i'(l_i)) + \sigma_i. \end{aligned}$$

Since $\beta_i < \beta_j$, we have $x_i > x_j$, which implies that $x_i > 0$. Therefore, we obtain

$$Ra_i + f_i\big[ql_i + F_i(l_i)\big] = 0.$$

Hence, $a_i < 0$ and then $a_j > 0$ which implies that $x_j = 0$. The impatient agent borrows from the patient agent.

We consider the case where $F_i(l_i) = A_i l_i^{\alpha}, F_j(l_i) = A_j l_j^{\alpha}$. Then $F_h'(l_h) = \alpha A_h l_h^{\alpha-1}$ for each $h = i, j$. In this case, we have $l_i, l_j > 0$, hence $\sigma_i = \sigma_j = 0$.

We see that $a_i < 0$, which implies that $a_j > 0$. Hence, $x_j = 0$. The asset price is $R = \dfrac{1}{\beta_j} = 1+r$, where r is the real interest rate. We have $q\Big(\frac{1}{\beta_j} - 1\Big) = F_i'(l_i) = \alpha A_j l_j^{\alpha-1}$, therefore

$$l_j = \Big(\frac{\alpha A_j}{\frac{1}{\beta_j} - 1}\frac{1}{q}\Big)^{\frac{1}{1-\alpha}}.$$

Since $\beta_i + x_i = \beta_j + x_j$, we get $x_i = \beta_j - \beta_i$. By consequence, we can compute

$$l_i = \Big(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1}\frac{1}{q}\Big)^{\frac{1}{1-\alpha}}.$$

Using $l_i + l_j = L$, we can compute the price of land

$$q^{\frac{1}{1-\alpha}}L = \Big(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1}\Big)^{\frac{1}{1-\alpha}} + \Big(\frac{\alpha A_j}{\frac{1}{\beta_j} - 1}\Big)^{\frac{1}{1-\alpha}}.$$

□

10 Appendix: Proofs for Section 4

Proof of Proposition 6. According to (26), it is easy to see that (i) is equivalent to (ii). We now prove that (ii) and (iii) are equivalent.

According to (25), we get that

$$q_0 = Q_T q_T \prod_{t=1}^{T}(1 + \frac{d_t}{q_t}).$$

Since $q_0 > 0$, we see that $\lim_{t\to+\infty} Q_t q_t > 0$ if and only if $\lim_{t\to\infty} \prod_{t=1}^{T}(1 + \frac{d_t}{q_t}) < \infty$. It is easy to prove that this condition is equivalent to $\sum_{t=1}^{\infty}\dfrac{d_t}{q_t} < +\infty$. □

Proof of Proposition 7. Assume that $Q_t/Q_{i,t}$ is uniformly bounded from above. According to Proposition 2, we have $\lim_{t\to\infty} Q_{i,t}(q_t l_{i,t} + a_{i,t}) = 0$, therefore

$$\lim_{t\to\infty} Q_t(q_t l_{i,t} + a_{i,t}) = \lim_{t\to\infty}\Big(\frac{Q_t}{Q_{i,t}}\Big)\Big(Q_{i,t}(q_t l_{i,t} + a_{i,t})\Big) = 0$$

for any i. Note that $\sum_i l_{i,t} = L > 0$ and $\sum_i a_{i,t} = 0$ for any t, we obtain that $\lim_{t\to\infty} Q_t q_t = 0$ □

Proof of Corollary 5. Since $\mu_{i,t+1} = 0$ for every $t \geq T$, we have $\lambda_{i,t} = \lambda_{i,t+1}R_{t+1}$ for every $t \geq T$. By consequence, $\gamma_{i,t} = \gamma_t$ for any $t \geq T+1$. This implies that $Q_t/Q_{i,t}$ is uniformly bounded from above. According to Proposition 7, there is no bubble. □

Proof of Lemma 5. According to (26), we get $\sum_{t=0}^{\infty} Q_t d_t < \infty$. However we have $d_t > F_i'(L) > \min_i F_i'(L) > 0$ for every t. Therefore, we obtain $\sum_{t=0}^{\infty} Q_t < \infty$. Since $\sup_{i,t} e_{i,t} < \infty$ and $F_i(l_{i,t}) \leq F_i(L)$ for every i, t, we obtain that $\sum_{t=0}^{\infty} Q_t Y_t < \infty$. □

Proof of Lemma 6. We will claim that $\sup_{i,t}(Q_t a_{i,t}) < \infty$. Indeed, (4) is rewritten as

$$Q_{t+1}R_{t+1}a_{i,t} + f_i Q_{t+1}\Big(q_{t+1}l_{i,t} + F_i(l_{i,t})\Big) \geq 0. \tag{98}$$

Since $Q_t = R_{t+1}Q_{t+1}$, (4) is equivalent to

$$Q_t a_{i,t} \geq -f_i Q_{t+1}\Big(q_{t+1}l_{i,t} + F_i(l_{i,t})\Big). \tag{99}$$

It is easy to see that $0 \leq Q_t q_t l_{i,t-1} \leq q_0 L < \infty$. Therefore, we have

$$f_i Q_{t+1}\Big(q_{t+1}l_{i,t} + F_i(l_{i,t})\Big) \leq f_i q_0 L + f_i Q_{t+1}F_i(L). \tag{100}$$

By consequence, we obtain

$$Q_t a_{i,t} \geq -f_i q_0 L - f_i Q_{t+1}F_i(L). \tag{101}$$

According to the proof of Lemma 5 , we see that $\lim_{t\to\infty} Q_t = 0$, and hence we get that $\inf_{i,t} Q_t a_{i,t} > -\infty$. Since $\sum_{i=1}^{m} Q_t a_{i,t} = 0$, we have $-\infty < \inf_{i,t} Q_t a_{i,t} \leq \sup_{i,t} Q_t a_{i,t} < \infty$. □

Proof of Lemma 7. We rewrite the budget constraint of agent i at date t as follows

$$Q_t c_{i,t} + Q_t q_t l_{i,t} + Q_t a_{i,t} = Q_t(e_{i,t} + F_i(l_{i,t-1})) + Q_t q_t l_{i,t-1} + Q_t R_t a_{i,t-1}. \tag{102}$$

According to (16) and (20), we get

$$Q_t q_t = Q_{t+1}\big(q_{t+1} + d_{t+1}\big), \quad Q_t = R_{t+1}Q_{t+1}.$$

Therefore, we have

$$\sum_{t=0}^{T} Q_t c_{i,t} + \sum_{t=1}^{T} Q_t d_t l_{i,t-1} + Q_T(q_T l_{i,T} + a_{i,T}) = \sum_{t=0}^{T} Q_t(e_{i,t} + F_i(l_{i,t-1})) + q_0 l_{i,-1} + R_0 a_{i,-1}.$$

By combining this with Lemmas 5 and 6, we obtain that

$$\sup_{T\to\infty}\Big(\sum_{t=0}^{T} Q_t c_{i,t} + \sum_{t=1}^{T} Q_t d_t l_{i,t-1}\Big) < \infty.$$

This implies that there exists the sum $\sum_{t=0}^{\infty}(Q_t c_{i,t} + Q_t d_t l_{i,t-1})$, and so does $\lim_{t\to\infty} Q_t\big(q_t l_{i,t} + a_{i,t}\big)$.

Note that $\lim_{t\to\infty} Q_t c_{i,t} = \lim_{t\to\infty} Q_t(e_{i,t} + F_i(l_{i,t-1})) = 0$. Then, by using (102), we get (27). □

Proof of Lemma 8. If $\lim_{t\to\infty} Q_t\big(a_{i,t} + q_t l_{i,t}\big) > 0$, there exists $T_1 \geq T$ such that $Q_t\big(a_{i,t} + q_t l_{i,t}\big) > 0$ for every $t \geq T_1$. Hence, we get

$$\begin{aligned}&Q_{t+1}R_{t+1}a_{i,t} + f_i Q_{t+1}\big[q_{t+1}l_{i,t} + F_i(l_{i,t})\big]\\ &\geq Q_{t+1}R_{t+1}a_{i,t} + Q_{t+1}\big[q_{t+1} + d_{t+1}\big]l_{i,t} = Q_{t+1}R_{t+1}a_{i,t} + Q_t q_t l_{i,t} > 0\end{aligned}$$

for every $t \geq T_1$. This implies that $\mu_{i,t+1} = 0$ for every $t \geq T_1$

Therefore, $\lambda_{i,t} = \lambda_{i,t+1}R_{t+1}$ for every $t \geq T_1$. By consequence, there exists a constant $C_i > 0$ such that $Q_t = C_i\lambda_{i,t}$ for every $t \geq T_1$. According to transversality condition (14), we get $\lim_{t\to\infty} Q_t\big(a_{i,t} + q_t l_{i,t}\big) = 0$, contradiction. □

Proof of Propostion 8. If $l_{i,t} = 0$, then condition (28) is satisfied.

If $l_{i,t} > 0$, by combining with $f_i = 1$ and using Lemma 3, we have $d_t = F_i'(l_{i,t-1}) \leq \frac{F_i(l_{i,t-1})}{l_{i,t-1}}$. Therefore, condition (28) is satisfied. By consequence, we have $\lim_{t\to\infty} Q_t\big(a_{i,t} + q_t l_{i,t}\big) \leq 0$ for any i. By summing this inequality over i, we obtain $\lim_{t\to\infty} Q_t q_t L \leq 0$, which implies that bubbles are ruled out. □

Proof of Proposition 10. 1. Since $Q_t \geq Q_{i,t}$, it is easy to see that $FV_0 \leq FV_i$ for any i, and if i-land bubbles exist for some agent i then land bubbles exist.

2. Assume that i-bubble exists for any t, we have $\lim_{t\to\infty} Q_{i,t} q_t > 0$ for any i. Therefore, we get $\lim_{t\to\infty} Q_t q_t > 0$. Since both these two limits are finite (less than q_0) and strictly positive, we see that $\lim_{t\to\infty} Q_t/Q_{i,t} \in (0,\infty)$ for any i. According to Proposition 7 we have $\lim_{t\to\infty} Q_t q_t = 0$, contradiction.

3. We now assume that $FV_0 = FV_i$ for any i, which implies that $\lim_{t\to\infty} Q_t q_t = \lim_{t\to\infty} Q_{i,t} q_t$. If land bubbles exist, we have $\lim_{t\to\infty} Q_t q_t = \lim_{t\to\infty} Q_{i,t} q_t \in (0, q_0)$. Thus, we obtain $\lim_{t\to\infty} Q_{i,t}/Q_t = 1$. According to Proposition 7 we have $\lim_{t\to\infty} Q_t q_t = 0 = \lim_{t\to\infty} Q_{i,t} q_t$, contradiction. □

11 Appendix: Proofs for Section 5

First, we give sufficient conditions for a sequence $\big(q_t, (c_{i,t}, l_{i,t})_{i\in I}\big)_t$ to be an equilibrium. Notice that the utility function may satisfy $u_i(0) = -\infty$.

Lemma 9. *If a sequence $\big(q_t, (c_{i,t}, l_{i,t}, \mu_{i,t})_{i\in I}\big)_t$ satisfies the following conditions*

(i) $\forall t,\ \forall i,\ c_{i,t} > 0, l_{i,t} \geqslant 0,\ \mu_{i,t} \geqslant 0.\ \forall t, q_t > 0,$

(ii) first-order conditions:

$$q_t = \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}\big(q_{t+1} + F_{i,t}'(l_{i,t})\big) + \eta_{i,t}, \quad \eta_{i,t} l_{i,t} = 0, \tag{103}$$

(iii) transversality conditions: $\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t}) q_t l_{i,t} = 0$ *for any i,*

(iv) $c_{i,t} + q_t l_{i,t} = e_{i,t} + q_t l_{i,t-1} + F_{i,t}(l_{i,t-1}),$

(vi) $\sum_{i\in I} l_{i,t} = L,$

then the sequence $\big(q_t, (c_{i,t}, l_{i,t})_{i\in I}\big)_t$ is an equilibrium for the economy without financial market.

Proof. Using the same argument in the proof of Proposition 2. □

Check of the example in Section 5.1.1. We now check all conditions in Lemma 9. It is easy to see that the market clearing conditions are satisfied.

Let us check FOCs:

$$q_{2t} = \frac{\beta_B u_B'(c_{B,2t+1})}{u_B'(c_{B,2t})}\big(q_{2t+1} + B_{2t+1}\big) \geq \frac{\beta_A u_A'(c_{A,2t+1})}{u_A'(c_{A,2t})}(q_{2t+1} + A_{2t+1}) \tag{104}$$

$$q_{2t-1} = \frac{\beta_A u_A'(c_{A,2t})}{u_A'(c_{A,2t-1})}(q_{2t} + A_{2t}) \geq \frac{\beta_B u_B'(c_{B,2t})}{u_B'(c_{B,2t-1})}(q_{2t} + B_{2t}). \tag{105}$$

The equality in (104) is satisfied since

$$\frac{\beta_B u'_B(c_{B,2t+1})}{u'_B(c_{B,2t})}\big(q_{2t+1}+B_{2t+1}\big) = \frac{\beta_B(e_{B,2t}-q_{2t})}{q_{2t+1}+B_{2t+1}}\big(q_{2t+1}+B_{2t+1}\big) \tag{106}$$

$$= \beta_B(e_{B,2t}-q_{2t}) = q_{2t}. \tag{107}$$

We now prove the inequality in (104). We have

$$\frac{\beta_A u'_A(c_{A,2t+1})}{u'_A(c_{A,2t})}(q_{2t+1}+A_{2t+1}) = \frac{\beta_A(q_{2t}+A_{2t})}{e_{A,2t+1}-q_{2t+1}}(q_{2t+1}+A_{2t+1}) \tag{108}$$

$$= \frac{\beta_A(\frac{\beta_B}{1+\beta_B}e_{B,2t}+A_{2t})}{\frac{1}{1+\beta_A}e_{A,2t+1}}(\frac{\beta_A}{1+\beta_A}e_{A,2t+1}+A_{2t+1}) \tag{109}$$

By consequence, the inequality in (104) is equivalent to

$$\beta_A(\frac{\beta_B e_{B,2t}}{1+\beta_B}+A_{2t})(\frac{\beta_A e_{A,2t+1}}{1+\beta_A}+A_{2t+1}) \leq \beta_B\frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t+1}}{1+\beta_A} \tag{110}$$

which is the condition (32).

We have

$$\frac{\beta_B u'_B(c_{B,2t})}{u'_B(c_{B,2t-1})}(q_{2t}+B_{2t}) = \frac{\beta_B(q_{2t-1}+B_{2t-1})}{e_{B,2t}-q_{2t}}(q_{2t}+B_{2t}) \tag{111}$$

$$= \frac{\beta_B(\frac{\beta_A}{1+\beta_A}e_{A,2t-1}+B_{2t-1})}{\frac{1}{1+\beta_B}e_{B,2t}}(\frac{\beta_B}{1+\beta_B}e_{B,2t}+B_{2t}) \tag{112}$$

By consequence, the inequality in (105) is equivalent to

$$\beta_B(\frac{\beta_A e_{A,2t-1}}{1+\beta_A}+B_{2t-1})(\frac{\beta_B e_{B,2t}}{1+\beta}+B_{2t}) \leq \beta_A\frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t-1}}{1+\beta_A} \tag{113}$$

which is the condition (33).

We now check TVCs. We have

$$\beta_A^{2t}u'_A(c_{A,2t})q_{2t}l_{A,2t+1} = 0 \tag{114}$$

$$\beta_A^{2t-1}u'_A(c_{A,2t-1})q_{2t-1}l_{A,2t} = \frac{\beta_A^{2t-1}}{c_{A,2t-1}}q_{2t-1} = \beta_A^{2t} \to 0. \tag{115}$$

Similarly, we also have

$$\beta_B^{2t}u'_B(c_{B,2t})q_{2t}l_{B,2t+1} = \beta_B^{2t+1} \to 0 \tag{116}$$

$$\beta_B^{2t-1}u'_B(c_{B,2t-1})q_{2t-1}l_{B,2t} = 0. \tag{117}$$

We finally verify that, for each $t \geq 0$,

$$\frac{\beta_B u'_B(c_{B,2t+1})}{u'_B(c_{B,2t})} \geq \frac{\beta_A u'_A(c_{A,2t+1})}{u'_A(c_{A,2t})} \tag{118}$$

$$\frac{\beta_B u'_B(c_{B,2t})}{u'_B(c_{B,2t-1})} \leq \frac{\beta_A u'_A(c_{A,2t})}{u'_A(c_{A,2t-1})}. \tag{119}$$

Indeed, condition (118) is rewritten as

$$\frac{\beta_B(e_{B,2t}-q_{2t})}{q_{2t+1}+B_{2t+1}} \geq \frac{\beta_A(\frac{\beta_B}{1+\beta_B}e_{B,2t}+A_{2t})}{\frac{1}{1+\beta_A}e_{A,2t+1}}. \tag{120}$$

Since $q_{2t} = \frac{\beta_B}{1+\beta_B}e_{B,2t}, q_{2t+1} = \frac{\beta_A}{1+\beta_A}e_{A,2t+1}$, condition (118) is equivalent to

$$\beta_A(\frac{\beta_B e_{B,2t}}{1+\beta_B} + A_{2t})(\frac{\beta_A e_{A,2t+1}}{1+\beta_A} + B_{2t+1}) \leq \beta_B \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t+1}}{1+\beta_A}. \tag{121}$$

This is condition (34).

By the same argument, we see that condition (119) is equivalent to

$$\beta_B(\frac{\beta_A e_{A,2t-1}}{1+\beta_A} + B_{2t-1})(\frac{\beta_B e_{B,2t}}{1+\beta} + A_{2t}) \leq \beta_A \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t-1}}{1+\beta_A}. \tag{122}$$

This is condition (35).

□

Check for the example in Section 5.2. We will find equilibria such that

$$(l_{A,2t-1}, l_{B,2t-1}) = (0,1), \quad (l_{A,2t}, l_{B,2t}) = (1,0) \tag{123}$$

$$a_{A,2t-1} = \frac{q_{2t} + B_{2t}}{R_{2t}} = -a_{B,2t-1}, \quad a_{B,2t} = \frac{q_{2t+1} + A_{2t+1}}{R_{2t+1}} = -a_{A,2t} \quad \forall t \geq 1. \tag{124}$$

It means that at any even (odd) date, agent A (agent B) borrows until her borrowing constraint is binding and buys land.[19] In this case, we have

$$\begin{aligned}
\forall t \geq 0, \quad & c_{A,2t} + q_{2t} + a_{A,2t} = e_{A,2t} + R_{2t}a_{A,2t-1} \\
\forall t \geq 0, \quad & c_{A,2t+1} + a_{A,2t+1} = e_{A,2t+1} \\
& c_{B,0} + a_{B,0} = e_{B,0} + q_0 + B_0 \\
\forall t \geq 1, \quad & c_{B,2t} + a_{B,2t} = e_{B,2t} \\
\forall t \geq 0, \quad & c_{B,2t+1} + q_{2t+1} + a_{B,2t+1} = e_{B,2t+1} + R_{2t+1}a_{B,2t}
\end{aligned}$$

Since $a_{B,2t} > 0$ and $a_{A,2t-1} > 0$, we have $\mu_{B,2t+1} = \mu_{A,2t} = 0$. Since agent A produces at date $2t+1$ and agent B produces at date $2t$, we have $\eta_{A,2t} = \eta_{B,2t-1} = 0$.

The consumption good is taken as numéraie $p_t = 1$. We have to find land prices and interest rates satisfying first-order and transversality conditions.

Transversality conditions (14) are written $\lim_{t\to\infty} \beta^t u_i'(c_{i,t})(q_t l_{i,t} + a_{i,t}) = 0$ for any i, or equivalently

$$\lim_{t\to\infty} \frac{\beta^{2t}(q_{2t} - \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}})}{e_{A,2t} + (q_{2t} + B_{2t}) - q_{2t} + \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}} = 0, \quad \lim_{t\to\infty} \frac{\beta^{2t+1}\frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}}}{e_{A,2t+1} - \frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}}} = 0$$

$$\lim_{t\to\infty} \frac{\beta^{2t}\frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}}{e_{B,2t} - \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}} = 0, \quad \lim_{t\to\infty} \frac{\beta^{2t+1}(q_{2t+1} - \frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}})}{e_{B,2t+1} + (q_{2t+1} + A_{2t+1}) - q_{2t+1} + \frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}}} = 0.$$

[19] We need this because Corollary 6 indicates that bubbles only exist if borrowing constraints of agents are frequently binding.

FOCs can be rewritten as

$$1 = \beta \frac{e_{B,0} + q_0 + B_0 - \frac{q_1+A_1}{R_1}}{e_{B,1} + (q_1 + A_1) - q_1 + \frac{q_2+B_2}{R_2}} R_1 > \beta \frac{e_{A,0} - q_0 + \frac{q_1+A_1}{R_1}}{e_{A,1} - \frac{q_2+B_2}{R_2}} R_1 \tag{125}$$

$$\begin{aligned} 1 &= \beta \frac{e_{B,2t} - \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}}{e_{B,2t+1} + (q_{2t+1} + A_{2t+1}) - q_{2t+1} + \frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}}} R_{2t+1} \\ &> \beta \frac{e_{A,2t} + (q_{2t} + B_{2t}) - q_{2t} + \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}}{e_{A,2t+1} - \frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}}} R_{2t+1} \end{aligned} \tag{126}$$

$$\begin{aligned} 1 &= \beta \frac{e_{A,2t-1} - \frac{q_{2t}+B_{2t}}{R_{2t}}}{e_{A,2t} + (q_{2t} + B_{2t}) - q_{2t} + \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}} R_{2t} \\ &> \beta \frac{e_{B,2t-1} + (q_{2t-1} + A_{2t-1}) - q_{2t-1} + \frac{q_{2t}+B_{2t}}{R_{2t}}}{e_{B,2t} - \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}} R_{2t}. \end{aligned} \tag{127}$$

Condition (125) means that $1 = \gamma_{B,1}R_1 > \gamma_{A,1}R_1$. Condition (126) means that $1 = \gamma_{B,2t+1}R_{2t+1} > \gamma_{A,2t+1}R_{2t+1}$ while condition (127) means that $1 = \gamma_{A,2t}R_{2t} > \gamma_{B,2t}R_{2t}$. This implies that $\gamma_{2t} = \gamma_{A,2t}$ and $\gamma_{2t+1} = \gamma_{B,2t+1}$.

It is easy to check that in our example, all first-order and transversality conditions are satisfied. □

12 Appendix: Proofs for Section 6

Proof of Proposition 12. We keep the notation p_ξ because our approach can also be applied when there are many consumption goods. Here, we allow for non-stationary production function $F_{i,\xi}$.

Point 1. The proof is far from trivial. Here, we present a proof, inspired by that in Araujo et al. (2011), Pascoa et al. (2011), which is different from that in the deterministic case.

Let $\Big(p, q, R, (c_i, l_i, a_i)_{i=1}^m\Big)$ be an equilibrium. For each agent i, we define $T-$horizon truncated optimization problem

$$P_i^T(p,q,R): \quad \max_{(c_i,l_i,a_i)\in B_i^T(p,q,R)} \Big[\sum_{\xi\in D^T} u_{i,\xi}(c_{i,\xi})\Big]$$

where

$$\begin{aligned} B_i^T(p,q,R) := \Big\{&(c_i,l_i,a_i) = (c_{i,\xi}, l_{i,\xi}, b_{i,\xi})_{\xi\in D^T} : \text{(a) } l_{i,\xi} = a_{i,\xi} = 0 \ \forall \xi \in D_T, \\ &c_{i,\xi} \geq 0, l_{i,\xi} \geq 0 \ \forall \xi \in D^T, \\ &\text{(b) } p_{\xi_0}c_{i,\xi_0} + q_{\xi_0}l_{i,\xi_0} + p_{\xi_0}a_{i,\xi_0} \leq p_{\xi_0}e_{i,\xi_0} + p_{\xi_0}F_{i,\xi_0}(l_{i,\xi_0^-}) + q_{\xi_0}l_{i,\xi_0^-} \\ &\text{(c) for each } \xi : 1 \leq t(\xi) \leq T : \\ &\quad 0 \leq R_\xi a_{i,\xi^-} + f_i\Big(q_\xi l_{i,\xi^-} + p_\xi F_{i,\xi}(l_{i,\xi^-})\Big) \\ &\quad p_\xi c_{i,\xi} + q_\xi l_{i,\xi} + p_\xi a_{i,\xi} \leq p_\xi e_{i,\xi} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + q_\xi l_{i,\xi^-} + R_\xi a_{i,\xi^-}\Big\}. \end{aligned}$$

If we define as in Araujo et al. (2011), because of the presence of short-sales, it will be not easy to prove that $B_i^T(p,q,R)$ is bounded. Unlike Araujo et al. (2011), we require $l_{i,\xi} = a_{i,\xi} = 0 \ \forall \xi \in D_T$. This trick helps us to prove more easily the boundedness of $B_i^T(p,q,R)$.

Lemma 10. *The problem $P_i^T(p,q,R)$ has a solution.*

Proof. It is easy to see that $\sup_{(c_i,l_i,a_i)\in B_i^T(p,q,R)}\Big[\sum_{\xi\in D^T} u_{i,\xi}(c_{i,\xi})\Big] < U_i(c_i) < \infty$ because of the optimality of (c_i, l_i, a_i) and condition (51). So, it implies that $B_i^T(p,q,R)$ is bounded. Indeed, fixe a node ξ: quantity of land bought by agent i at node ξ^- is bounded because otherwise $p_\xi F_{i,\xi}(l_{\xi^-}) + q_\xi l_{\xi^-} = \infty$, and then we can choose $(c,l,a) \in B_i^T(p,q,R)$ such that $a_\mu = 0$ for any $\mu \in D_T$ and $c_\xi = \infty$ which gives an infinity utility, a contradiction. Asset volumes are bounded from below thanks to borrowing constraints and the boundedness of land holding; they are bounded from above because of budget constraints. Thus, it is easy to see that (c_i) is bounded. Therefore, $B_i^T(p,q,R)$ is bounded and so compact. This implies that the problem $P_i^T(p,q,R)$ has a solution. □

Let (c_i^T, l_i^T, a_i^T) be a solution of the problem $P_i^T(p,q,R)$

The Langrangian of the problem $P_i^T(p,q,R)$ is[20]

$$L^T(c,l,a,\lambda_i,\eta_i,\mu_i) = \sum_{\xi\in D^T} u_{i,\xi}(c_\xi) + \sum_{\xi\in D^{T-1}} \eta_{i,\xi}^T l_\xi \tag{128}$$

$$+ \sum_{\xi\in D^T\setminus\{\xi_0\}} \mu_{i,\xi}^T\Big(R_\xi a_{\xi^-} + f_i\big(q_\xi l_{\xi^-} + p_\xi F_{i,\xi}(l_{\xi^-})\big)\Big)$$

$$+ \sum_{\xi\in D^{T-1}} \lambda_{i,\xi}^T\Big(p_\xi e_{i,\xi} + p_\xi F_{i,\xi}(l_{\xi^-}) + q_\xi l_{\xi^-} + R_\xi a_{\xi^-} - p_\xi c_\xi - q_\xi l_\xi - p_\xi a_\xi\Big)$$

$$+ \sum_{\xi\in D_T} \lambda_{i,\xi}^T\Big(p_\xi e_{i,\xi} + p_\xi F_{i,\xi}(l_{\xi^-}) + q_\xi l_{\xi^-} + R_\xi a_{\xi^-} - p_\xi c_\xi\Big)\Big]. \tag{129}$$

By using Kuhn-Tucker Theorem and the duality in convex programming in Florenzano and Le Van (2001) (Chapter 7) and Rockafellar (1997) (Chapter 28), there exist non-negative sequences

$$(\lambda_{i,\xi}^T)_{\xi\in D^T}, (\eta_{i,\xi}^T)_{\xi\in D^{T-1}}, (\mu_{i,\xi}^T)_{\xi\in D^T : t(\xi)\geq 1}$$

such that

$$L^T(c,l,a,\lambda_i^T,\eta_i^T,\mu_i^T) \leq L^T(c_i^T, l_i^T, a_i^T, \lambda_i^T, \eta_i^T, \mu_i^T) = \sum_{\xi\in D^T} u_{i,\xi}(c_{i,\xi}^T) \leq \sum_{\xi\in\mathcal{D}} u_{i,\xi}(c_{i,\xi}) \tag{130}$$

for any (c,l,a), where the last inequality in (130) is from the optimality of (c_i, l_i, a_i) for the original problem $(P_i(p,q,R))$. This implies that

$$\sum_{\xi\in D^T} u_{i,\xi}(c_\xi) + \sum_{\xi\in D^{T-1}} \eta_{i,\xi}^T l_\xi$$

$$+ \sum_{\xi\in D^T\setminus\{\xi_0\}} \mu_{i,\xi}^T\Big(R_\xi a_{\xi^-} + f_i\big(q_\xi l_{\xi^-} + p_\xi F_{i,\xi}(l_{\xi^-})\big)\Big)$$

$$+ \sum_{\xi\in D^{T-1}} \lambda_{i,\xi}^T\Big(p_\xi e_{i,\xi} + p_\xi F_{i,\xi}(l_{\xi^-}) + q_\xi l_{\xi^-} + R_\xi a_{\xi^-} - p_\xi c_\xi - q_\xi l_\xi - p_\xi a_\xi\Big)$$

$$+ \sum_{\xi\in D_T} \lambda_{i,\xi}^T\Big(p_\xi e_{i,\xi} + p_\xi F_{i,\xi}(l_{\xi^-}) + q_\xi l_{\xi^-} + R_\xi a_{\xi^-} - p_\xi c_\xi\Big)\Big]$$

$$\leq \sum_{\xi\in\mathcal{D}} u_{i,\xi}(c_{i,\xi}) \tag{131}$$

for any (c,l,a).

By choosing $c = l = a = 0$ in (131), we have, for any $t \leq T$,

$$\sum_{\xi\in D^t} \lambda_{i,\xi}^T p_\xi e_{i,\xi} \leq \sum_{\xi\in\mathcal{D}} u_{i,\xi}(c_{i,\xi}) < \infty. \tag{132}$$

[20] We omit multiplier associated to constraint $c_{i,\xi} \geq 0$ because at optimal we always have $c_{i,\xi} > 0$ thanks to Inada condition.

This implies that $(\lambda^T_{i,\xi})_{T\geq T(\xi)}$ is bounded node by node.

In condition (131), by choosing $(c,l,a)\in B^T_i(p,q,R)$ such that $l_\xi = l_{i,\xi_0^-}$ $\forall \xi$, we have, for any $t\leq T$,

$$\sum_{\xi\in D^{t-1}}\eta^T_{i,\xi}l_{i,\xi_0^-} + \sum_{\xi\in D^t} f_i\mu^T_{i,\xi}(q_\xi l_{i,\xi_0^-} + p_\xi F_i(l_{i,\xi_0^-})) \leq \sum_{\xi\in\mathcal{D}} u_{i,\xi}(c_{i,\xi}) < \infty. \tag{133}$$

Since $l_{i,\xi_0^-} > 0$, we obtain that $(\eta^T_{i,\xi},\mu^T_{i,\xi})_{T\geq t(\xi)+1}$ are bounded node by node. So there is a subsequence $(T_k)_k$ and a non-negative sequence $(\lambda_{i,\xi},\eta_{i,\xi},\mu_{i,\xi})$ such that, for each node ξ,

$$(\lambda^{T_k}_{i,\xi},\eta^{T_k}_{i,\xi},\mu^{T_k}_{i,\xi}) \longrightarrow (\lambda_{i,\xi},\eta_{i,\xi},\mu_{i,\xi}) \text{ when } k\to\infty.$$

We fixe $\xi\in D^T\setminus D_T$. Applying condition (131) for (c,l,a) such that $(c_\mu,l_\mu,a_\mu)=(c_{i,\mu},l_{i,\mu},a_{i,\mu})$ for any $\mu\neq\xi$, we have

$$\begin{aligned}
&u_{i,\xi}(c_\xi) + \eta^T_{i,\xi}l_\xi + \lambda^T_{i,\xi}\Big(p_\xi c_{i,\xi} + q_\xi l_{i,\xi} + p_\xi a_{i,\xi} - p_\xi c_\xi - q_\xi l_\xi - p_\xi a_\xi\Big)\\
&+\sum_{\xi'\in\xi^+}\mu^T_{i,\xi'}\Big(R_{\xi'}a_\xi + f_i\big(q_{\xi'}l_\xi + p_{\xi'}F_{i,\xi'}(l_\xi)\big)\Big)\\
&+\sum_{\xi'\in\xi^+}\lambda^T_{i,\xi'}\Big(p_{\xi'}F_{i,\xi'}(l_\xi) + q_{\xi'}l_\xi + R_{\xi'}a_\xi - p_{\xi'}F_{i,\xi'}(l_{i,\xi}) - q_{\xi'}l_{i,\xi} - R_{\xi'}a_{i,\xi}\Big)\\
&\leq u_{i,\xi}(c_{i,\xi}) + \sum_{\xi\in\mathcal{D}\setminus D^T} u_{i,\xi}(c_{i,\xi})
\end{aligned}\tag{134}$$

Now, let k in the subsequence (T_k) tend to infinity, we obtain that, for any (c_ξ,l_ξ,a_ξ),

$$\begin{aligned}
&u_{i,\xi}(c_\xi) + \eta_{i,\xi}l_\xi + \lambda_{i,\xi}\Big(p_\xi c_{i,\xi} + q_\xi l_{i,\xi} + p_\xi a_{i,\xi} - p_\xi c_\xi - q_\xi l_\xi - p_\xi a_\xi\Big)\\
&+\sum_{\xi'\in\xi^+}\mu_{i,\xi'}\Big(R_{\xi'}a_\xi + f_i\big(q_{\xi'}l_\xi + p_{\xi'}F_{i,\xi'}(l_\xi)\big)\Big)\\
&+\sum_{\xi'\in\xi^+}\lambda_{i,\xi'}\Big(p_{\xi'}F_{i,\xi'}(l_\xi) + q_{\xi'}l_\xi + R_{\xi'}a_\xi - p_{\xi'}F_{i,\xi'}(l_{i,\xi}) - q_{\xi'}l_{i,\xi} - R_{\xi'}a_{i,\xi}\Big)\\
&\leq u_{i,\xi}(c_{i,\xi}).
\end{aligned}\tag{135}$$

Choosing $(c_\xi,l_\xi,a_\xi)=(c_{i,\xi},l_{i,\xi},a_{i,\xi})$, we have

$$\eta_{i,\xi}l_{i,\xi} = \mu_{i,\xi'}\Big(R_{\xi'}a_{i,\xi} + f_i\big(q_{\xi'}l_{i,\xi} + p_{\xi'}F_{i,\xi'}(l_{i,\xi})\big)\Big) = 0 \ \forall\xi'\in\xi^+. \tag{136}$$

By using (135) and the definition of derivatives, we also obtain FOCs (52), (53), (54).

Let us now prove transversality condition (57).

Lemma 11. $\limsup\limits_{t\to\infty}\sum\limits_{\xi\in D_t}\lambda_{i,\xi}\big(q_\xi l_{i,\xi} + p_\xi a_{i,\xi}\big)\leq 0.$

Proof. Fixe t, and take $T>t$, we choose (c,l,a) such that $(c_\xi,l_\xi,a_\xi)=(c_{i,\xi},l_{i,\xi},a_{i,\xi})$ for any $\xi\in D^{t-1}$ and $(c_\xi,l_\xi,a_\xi)=0$ otherwise, condition (131) gives that

$$\sum_{\xi\in D_t}\lambda^T_{i,\xi}\Big(p_\xi e_{i,\xi} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + q_\xi l_{i,\xi^-} + R_\xi a_{i,\xi^-}\Big)\Big] \leq \sum_{\xi\in\mathcal{D}\setminus D^{t-1}} u_{i,\xi}(c_{i,\xi}) \tag{137}$$

for any T. Let k in the subsequence (T_k) tend to infinity, we have

$$\sum_{\xi\in D_t}\lambda_{i,\xi}\Big(p_\xi e_{i,\xi} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + q_\xi l_{i,\xi^-} + R_\xi a_{i,\xi^-}\Big)\Big] \leq \sum_{\xi\in\mathcal{D}\setminus D^{t-1}} u_{i,\xi}(c_{i,\xi}). \tag{138}$$

Let t tend to infinity, notice that $\sum_{\xi \in \mathcal{D}} u_{i,\xi}(c_{i,\xi}) < \infty$ and $u_{i,\xi}(c_{i,\xi}) \geq 0$, we obtain transversality condition

$$\lim_{t\to\infty} \sum_{\xi \in D_t} \lambda_{i,\xi}\Big(p_\xi e_{i,\xi} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + q_\xi l_{i,\xi^-} + R_\xi a_{i,\xi^-}\Big)\Big] = 0. \tag{139}$$

Combining this with the fact that budget constraints are always binding, we obtain

$$\limsup_{t\to\infty} \sum_{\xi \in D_t} \lambda_{i,\xi}\big(q_\xi l_{i,\xi} + p_\xi a_{i,\xi}\big) \leq 0. \tag{140}$$

□

We have, by noting that $u_{i,\xi}(0) \geq 0$,

$$\infty > \sum_{\xi \in \mathcal{D}} u_{i,\xi}(c_{i,\xi}) \geq \sum_{\xi \in \mathcal{D}} u_{i,\xi}(0) + \sum_{\xi \in \mathcal{D}} u'_{i,\xi}(c_{i,\xi})c_{i,\xi} \geq \sum_{\xi \in \mathcal{D}} \lambda_{i,\xi} p_\xi c_{i,\xi}. \tag{141}$$

So, we get $\lim_{t\to\infty} \sum_{\xi \in D_t} \lambda_{i,\xi} p_\xi c_{i,\xi} = 0$. By combining this with Lemma 11, condition (139), and the fact that budget constraints are always binding, we obtain (57) i.e., $\limsup_{t\to\infty} \sum_{\xi \in D_t} \lambda_{i,\xi}\big(q_\xi l_{i,\xi} + p_\xi a_{i,\xi}\big) = 0$.

Point 2. It is sufficient to prove the optimality of (c_i, l_i, a_i). This is proved by using the same argument of the deterministic case.

□

Proof of Proposition 14.
First, take γ in Assumption 11, we will prove that $(1-\gamma)(q_\xi l_{i,\xi} + a_{i,\xi}) \leq W_\xi$ for any i and for any ξ.[21] Indeed, suppose that there exist i and ξ such that $(1-\gamma)(q_\xi l_{i,\xi} + a_{i,\xi}) > W_\xi$. Let us consider a new allocation of agent i: $z_i := z\big(c_i, \xi, \gamma, (1-\gamma)(q_\xi l_{i,\xi} + a_{i,\xi})\big)$. By noticing that $\gamma < 1$ and $\gamma F(x) < F(\gamma x)$ $\forall x$, we can check that this allocation is in the budget set of agent i. By Assumption 11, we have

$$U_i(c_i) < U_i\Big(z(c_i, \xi, \gamma, W_\xi)\Big) < U_i\Big(z\big(c_i, \xi, \gamma, (1-\gamma)(q_\xi l_{i,\xi} + a_{i,\xi})\big)\Big). \tag{142}$$

This is in contradiction to the optimality of (c_i, l_i, a_i).

So, we have $(1-\gamma)(q_\xi l_{i,\xi} + a_{i,\xi}) \leq W_\xi$ $\forall i, \forall \xi$. Taking the sum over i, we get $(1-\gamma)q_\xi L \leq mW_\xi$ $\forall \xi$. Since $L(1-\gamma) > 0$, we get that

$$q_\xi \leq \frac{mW_\xi}{L(1-\gamma)} \ \forall \xi. \tag{143}$$

Point 1 of Proposition 14 is a direct consequence of (143).

We now prove point 2 of Proposition 14. Let us consider a process of state-price and land dividend $\Gamma := (\gamma_\xi, d_\xi)_{\xi \in \mathcal{D}}$. We will prove that $\lim_{t\to\infty} \sum_{\xi \in D_t} Q_\xi q_\xi = 0$.

We observe that $d_\xi \geq \inf_{i,\xi} F'_{i,\xi}(l_{i,\xi^-}) > \inf_{i,\xi} F'_{i,\xi}(L) > 0$ $\forall \xi$. Combining this with (66), we have

$$\infty > q_{\xi_0} \geq \sum_{t\geq 1} \sum_{\xi \in D_t} Q_\xi d_\xi \geq \Big(\inf_{i,\xi} F'_{i,\xi}(L)\Big) \sum_{t\geq 1} \sum_{\xi \in D_t} Q_\xi. \tag{144}$$

Therefore, $\lim_{t\to\infty} \sum_{\xi \in D_t} Q_\xi = 0$. Since W_ξ is uniformly bounded from above, we get that $\lim_{t\to\infty} \sum_{\xi \in D_t} Q_\xi W_\xi = 0$. According to point 1, we obtain $\lim_{t\to\infty} \sum_{\xi \in D_t} Q_\xi q_\xi = 0$.

□

[21] Our proof of this claim is inspired by that of Lemma 3.8 in Santos and Woodford (1997). Theorem 1's proof of Pascoa et al. (2011) uses a similar argument.

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Intertemporal equilibrium with heterogeneous agents, endogenous dividends and collateral constraints*
Online appendix

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Abstract

We provide a proof for the existence of equilibrium in a stochastic production economy with incomplete markets.

Keywords: Infinite-horizon, general equilibrium, collateral constraint, incomplete market.

JEL Classifications: C62, D53, D9, E44, G10.

1 Model

We follow the literature of infinite horizon incomplete markets as Magill and Quinzii (1994), Magill and Quinzii (1996), Kubler and Schmedders (2003), Magill and Quinzii (2008) and references therein, or more recently Araujo, Pascoa, and Torres-Martinez (2011), Pascoa, Petrassi and Torres-Martinez (2011).

Consider an infinite horizon discrete time economy where the set of dates is $0, 1, ...$ and there is no uncertainty at initial date ($t = 0$). Given a history of realizations of the states of nature for the first $t-1$ dates, with $t \geq 1$, $\bar{s}_t = (s_0, ..., s_{t-1})$, there is a finite set $\mathcal{S}(\bar{s}_t)$ of states that may occur at date t. A vector $\xi = (t, \bar{s}_t, s)$, where $t \geq 1$ and $s \in \mathcal{S}(\bar{s}_t)$, is called a *node*. The only node at $t = 0$ is denoted by ξ_0. Let $\mathcal{D}$ be the (countable) event-tree, i.e., the set of all nodes. We denote by $t(\xi)$ the date associated with a node ξ.

Given $\xi := (t, \bar{s}_t, s)$ and $\mu := (t', \bar{s}_{t'}, s')$, we say that μ is a successor of ξ, and we write $\mu > \xi$, if $t' > t$ and the first $t+1$ coordinates of $\bar{s}_{t'}$ are $(\bar{s}_t, s)$. We write $\mu \geq \xi$ to say that either $\mu > \xi$ or $\mu = \xi$.

For each T and ξ, we denote $D(\xi) := \{\mu : \mu \geq \xi\}$ the sub-tree with root ξ; $D_T := \{\xi : t(\xi) = T\}$ the family of nodes with date T; $D^T(\xi) := \bigcup_{t=t(\xi)}^{T} D_t(\xi)$, where $D_T(\xi) := D_T \cap D(\xi)$; $D^T := D^T(\xi_0)$; $\xi^+ := \{\mu \geq \xi : t(\mu) = t(\xi) + 1\}$ the set of immediate successors of ξ; ξ^- the unique predecessor of ξ.

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There is a single consumption good at each node. The number m of agents is finite. I denotes the set of agents. At each node ξ, each agent i is endowed $e_{i,\xi} > 0$ units of consumption good.

Each household i takes the sequence of prices $(p, q, R) := (p_\xi, q_\xi, R_\xi)_{\xi\in\mathcal{D}}$ as given and chooses sequences of consumption, land, and asset volume $(c_i, l_i, a_i) := (c_{i,\xi}, l_{i,\xi}, a_{i,\xi})_{\xi\in\mathcal{D}}$ in order to maximizes her intertemporal utility

$$P_i(p, q, R): \quad \max_{(c_i,l_i,a_i)} \Big[U_i(c_i) := \sum_{\xi\in\mathcal{D}} u_{i,\xi}(c_{i,\xi})\Big]$$

subject to, for each $\xi \geq \xi_0$,

$$l_{i,\xi} \geq 0 \tag{1}$$

$$p_\xi c_{i,\xi} + q_\xi l_{i,\xi} + p_\xi a_{i,\xi} \leq p_\xi e_{i,\xi} + q_\xi l_{i,\xi^-} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + R_\xi a_{i,\xi^-} \tag{2}$$

$$R_{\xi'} a_{i,\xi} \geq -f_i\big[q_{\xi'} l_{i,\xi} + p_{\xi'} F_{i,\xi'}(l_{i,\xi})\big] \quad \forall \xi' \in \xi^+, \tag{3}$$

where $l_{i,\xi_0^-} > 0$ is given and $a_{i,\xi_0^-} = 0$. Here, the production function $F_{i,\xi}$ of agent i depend on both i and node ξ.

The deterministic model corresponds to the case where $\mathcal{D} = \{0, 1, 2, \ldots\}$ and $u_{i,\xi}(c) = \beta_i^{t(\xi)} u_i(c)$.

Another particular case of our model, where $F_{i,\xi} = 0$, $f_i = 0\ \forall i, \forall \xi$, and there is no short-sale, corresponds to Pascoa, Petrassi and Torres-Martinez (2011). In this case, land can be interpreted as fiat money. However, Pascoa, Petrassi and Torres-Martinez (2011) assume that agents have money endowments at each node while we consider that agents have land endowments only at initial node.

The economy is denoted by $\mathcal{E}$ characterized by a list of fundamentals

$$\mathcal{E} := \Big((u_{i,\xi}, e_{i,\xi}, F_{i,\xi})_{\xi\in\mathcal{D}}, f_i, l_{i,\xi_0^-}, a_{i,\xi_0^-}\Big)_{i\in I}.$$

Definition 1. *Given the economy $\mathcal{E}$. A list $\Big(\bar{p}_\xi, \bar{q}_\xi, \bar{R}_\xi, (\bar{c}_{i,\xi}, \bar{l}_{i,\xi}, \bar{a}_{i,\xi})_{i=1}^m\Big)_{\xi\in\mathcal{D}}$ is an equilibrium if the following conditions are satisfied:*

(i) Price positivity: $\bar{p}_\xi, \bar{q}_\xi, \bar{R}_\xi > 0$ for any ξ.

(ii) Market clearing: at each ξ,

$$\textit{good: } \sum_{i=1}^m \bar{c}_{i,\xi} = \sum_{i=1}^m (e_{i,\xi} + F_{i,\xi}(\bar{l}_{i,\xi^-})) \tag{4}$$

$$\textit{land: } \sum_{i=1}^m \bar{l}_{i,\xi} = L \tag{5}$$

$$\textit{financial asset: } \sum_{i=1}^m \bar{a}_{i,\xi} = 0. \tag{6}$$

(iii) Agents' optimality: for each i, $(\bar{c}_{i,\xi}, \bar{l}_{i,\xi}, \bar{a}_{i,\xi})_{\xi\in\mathcal{D}}$ is a solution of the problem $P_i(\bar{p}, \bar{q}, \bar{R})$.

Note that the financial asset in our framework is a short-lived asset with zero supply, which is different from the long-lived asset bringing exogenous positive dividends in Lucas (1978), Santos and Woodford (1997), Le Van and Pham (2016). Instead, when production functions are given by $F_{i,\xi}(x) = d_\xi x\ \forall i, \forall \xi$, land in our model corresponds to this asset with exogenous dividends; in particular, when $F_{i,\xi} = 0\ \forall i, \forall \xi$, land becomes fiat money as in Bewley (1980) or pure bubble asset as in Tirole (1985).

Assumption 1 (production functions). *For each i, the function $F_{i,\xi}$ is concave, continuously differentiable, $F'_{i,\xi} > 0$, $F_{i,\xi}(0) = 0$.*

Assumption 2 (endowments). *$l_{i,\xi_0^-} > 0$ and $a_{i,\xi_0^-} = 0$ for any i. $e_{i,t} > 0$ for any i and for any t.*

Assumption 3 (borrowing limits). *$f_i \in (0, 1]$ for any i.*

Assumption 4 (utility functions). *For each i and for each $\xi \in \mathcal{D}$, the function $u_{i,\xi}$: $\mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable, concave, $u_{i,\xi}(0) = 0$, $u_{i,\xi}(+\infty) = +\infty$, $u'_{i,\xi} > 0$, $u'_{i,\xi}(0) = \infty$.*[1]

Assumption 5 (finite utility). *For each i,*

$$\sum_{\xi \in \mathcal{D}} u_{i,\xi}(W_\xi) < \infty, \qquad \text{where } W_\xi := \sum_{i=1}^{m} \Big(e_{i,\xi} + F_{i,\xi}(L)\Big). \tag{7}$$

Proposition 1. *Under assumptions (1) to (5), there exists an equilibrium.*

2 Equilibrium existence

We will provide a proof for Proposition 1. Notice that we cannot directly use a method of Becker, Bosi, Le Van, and Seegmuller (2015) or Le Van and Pham (2016) because the financial asset in our model is a short-lived asset with zero supply. The difficulty is to prove that the asset volume $a_{i,t}$ is bounded. To overcome this difficulty, we introduce an intermediate economy with a nominal asset whose structure is different from that of the financial asset in the original economy. In this intermediate economy, we can bound the volume of the financial asset, and so can prove the existence of equilibrium by adapting the method of Becker, Bosi, Le Van, and Seegmuller (2015) and Le Van and Pham (2016): (1) we prove the existence of equilibrium for each $T-$ truncated economy $\mathcal{E}^T$; (2) we show that this sequence of equilibria converges for the product topology to an equilibrium of our economy $\mathcal{E}$. Last, we construct an equilibrium for the original economy from an equilibrium of the intermediate economy.

Let us introduce the intermediate economy $\tilde{\mathcal{E}}$ as follows. The intermediate economy $\tilde{\mathcal{E}}$ is defined as the original economy $\mathcal{E}$ except the maximization problems of consumers are now

$$P_i(p, q, r): \quad \max_{(c_i, l_i, a_i)} \sum_{\xi \in \mathcal{D}} u_{i,\xi}(c_{i,\xi})$$

subject to, for each $\xi \geq \xi_0$,

$$l_{i,\xi} \geq 0 \tag{8}$$

$$p_\xi c_{i,\xi} + q_\xi l_{i,\xi} + b_{i,\xi} \leq p_\xi e_{i,\xi} + q_\xi l_{i,\xi^-} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + r_\xi b_{i,\xi^-} \tag{9}$$

$$r_{\xi'} b_{i,\xi} \geq -f_i \big[q_{\xi'} l_{i,\xi} + p_{\xi'} F_{i,\xi'}(l_{i,\xi})\big] \quad \forall \xi' \in \xi^+, \tag{10}$$

where $l_{i,\xi_0^-} > 0$ is given and $a_{i,\xi_0^-} = 0$.

To proof the existence of equilibrium for the original economy, it is sufficient to prove the existence of equilibrium for the economy $\tilde{\mathcal{E}}$.

[1] In the proof of the existence of equilibrium, we do not require $u'_{i,\xi}(0) = \infty$. This condition is to ensure that $c_{i,\xi} > 0$ for any ξ, which is used in next Sections.

2.1 Existence of equilibrium for truncated economies

For each $T \geq 1$, we define $T-$horizon truncated economy $\tilde{\mathcal{E}}^T$ as $\tilde{\mathcal{E}}$ but there are no activities from period $T+1$ to the infinity, i.e., $c_{i,\xi'} = l_{i,\xi} = b_{i,\xi} = 0$ for every $i = 1, \ldots, m$ and for any ξ and $\xi' \in \xi^+$ with $t(\xi) \geq T$.

We then define the bounded economy $\tilde{\mathcal{E}}^T_b$ as $\tilde{\mathcal{E}}^T$ but consumption level $(c_{i,\xi})_{\xi \in D^T}$, land holding $(l_{i,t})_{\xi \in D^{T-1}}$, and asset holding $(b_{i,t})_{\xi \in D^{T-1}}$ are respectively bounded in the following sets:

$$\mathcal{C} \;:=\; [0, B_c]^{|D^T|}, \quad \mathcal{L} := [0, B_l]^{|D^{T-1}|}, \quad \mathcal{A} := \prod_{t=1}^{T} [-B_b, B_b]^{|D^{T-1}|},$$

where $|S|$ denotes the cardinal of the set S, $B_c > \max_{\xi \in D^T} \sum_{i=1}^{m} \Big(e_{i,\xi} + F_{i,\xi}(B_l)\Big)$, $B_l > L$, and $B_b = m(B_c + B_l)$.

The economy $\tilde{\mathcal{E}}^T_b$ depends on bounds B_c, B_l, B_b, so we write $\tilde{\mathcal{E}}^T_b(B_c, B_l, B_b)$.

Let us denote

$$\mathcal{X}_b := \mathcal{C} \times \mathcal{L} \times \mathcal{A}, \quad \mathcal{X} := (\mathcal{X}_b)^m \tag{11}$$

We then define

$$\mathcal{P} := \{z_0 = (p, q, r) : r_{\xi_0} = 0, q_\xi = 0 \ \forall \xi \in D^T; \tag{12}$$

$$p_\xi, q_\xi, r_\xi \geq 0; \quad 2p_\xi + q_\xi + r_\xi = 1 \ \forall \xi \in D^T\} \tag{13}$$

$$\Phi := \mathcal{P} \times \mathcal{X}. \tag{14}$$

An element $z \in \Phi$ is in the form $z = (z_i)_{i=0}^m$ where $z_0 := (p, q, r), z_i := (c_i, l_i, b_i)$ for each $i = 1, \ldots, m$.

The following remark is to ensure that the asset volume $(b_{i,\xi})$ is bounded.

Remark 1. *If $z \in \Phi$ is an equilibrium for the economy $\tilde{\mathcal{E}}$ then $c_{i,\xi} \in [0, B_c)$, $l_{i,\xi} \in [0, L]$. By using the fact that $2p_\xi + q_\xi + r_\xi = 1$, we have $b_{i,\xi} \leq B_c + B_l$ for any i, ξ. Indeed, this is true for $\xi = \xi_0$ because*

$$b_{i,\xi_0} \leq p_{\xi_0} e_{i,\xi_0} + q_{\xi_0} l_{i,\xi_0^-} + p_{\xi_0} F_{i,\xi_0}(l_{i,\xi_0^-}) \leq (2p_{\xi_0} + q_{\xi_0})(B_c + B_l) = B_c + B_l \tag{15}$$

and then, for any $\xi \geq \xi_0$, we have

$$b_{i,\xi} \leq p_\xi e_{i,\xi} + q_\xi l_{i,\xi^-} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + r_\xi b_{i,\xi^-} \leq (2p_\xi + q_\xi + r_\xi)(B_c + B_l) = B_c + B_l \tag{16}$$

Since $\sum_{i=1}^{m} b_{i,\xi} = 0$, we get that $b_{i,\xi} \in [-B_b, B_b]$ for any i and any ξ.

Proposition 2. *Under Assumptions (1-4), there exists an equilibrium $(p, q, r, (c_i, l_i, b_i)_{i=1}^m)$, with $2p_\xi + q_\xi + r_\xi = 1 \ \forall \xi$, for the economy $\tilde{\mathcal{E}}^T_b(B_c, B_l, B_b)$.*[2]

[2]Notice that, in the definition of equilibrium for $\tilde{\mathcal{E}}^T_b(B_c, B_l, B_b)$, in period T, we only require $\sum_i (l_{i,\xi} - l_{i,\xi^-}) \leq 0$ and $\sum_i (l_{i,\xi} - l_{i,\xi^-}) q_\xi = 0 \ \forall \xi \in D_T$. These conditions are satisfied because we impose $q_\xi = 0$ and $l_{i,\xi} = 0$ for any $\xi \in D_T$.

Proof. We firstly define

$$\begin{aligned}
B_i^T(p,q,r) := \big\{ & (c_{i,\xi}, l_{i,\xi}, b_{i,\xi})_{\xi \in D^T} \in \mathcal{X} : \text{(a) } l_{i,\xi} = b_{i,\xi} = 0 \ \forall \xi \in D_T, \\
& \text{(b) } p_{\xi_0} c_{i,\xi_0} + q_{\xi_0} l_{i,\xi_0} + b_{i,\xi_0} \leq p_{\xi_0} e_{i,\xi_0} + p_{\xi_0} F_{i,\xi_0}(l_{i,\xi_0^-}) + q_{\xi_0} l_{i,\xi_0^-} \\
& \text{(c) for each } \xi : 1 \leq t(\xi) \leq T : \\
& \quad 0 \leq r_\xi b_{i,\xi^-} + f_i\Big(q_\xi l_{i,\xi^-} + p_\xi F_{i,\xi}(l_{i,\xi^-})\Big) \\
& \quad p_\xi c_{i,\xi} + q_\xi l_{i,\xi} + b_{i,\xi} \leq p_\xi e_{i,\xi} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + q_\xi l_{i,\xi^-} + r_\xi b_{i,\xi^-} \big\}.
\end{aligned}$$

We also define $C_i^T(p,q,r)$ as follows.

$$\begin{aligned}
C_i^T(p,q,r) := \big\{ & (c_{i,\xi}, l_{i,\xi}, b_{i,\xi})_{\xi \in D^T} \in \mathcal{X} : \text{(a) } l_{i,\xi} = b_{i,\xi} = 0 \ \forall \xi \in D_T, \\
& \text{(b) } p_{\xi_0} c_{i,\xi_0} + q_{\xi_0} l_{i,\xi_0} + b_{i,\xi_0} < p_{\xi_0} e_{i,\xi_0} + p_{\xi_0} F_{i,\xi_0}(l_{i,\xi_0^-}) + q_{\xi_0} l_{i,\xi_0^-} \\
& \text{(c) for each } \xi : 1 \leq t(\xi) \leq T : \\
& \quad 0 < r_\xi b_{i,\xi^-} + f_i\Big(q_\xi l_{i,\xi^-} + p_\xi F_{i,\xi}(l_{i,\xi^-})\Big) \\
& \quad p_\xi c_{i,\xi} + q_\xi l_{i,\xi} + b_{i,\xi} < p_\xi e_{i,\xi} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + q_\xi l_{i,\xi^-} + r_\xi b_{i,\xi^-} \big\}.
\end{aligned}$$

Lemma 1. *$C_i^T(p,q,r) \neq \emptyset$ and $\bar{C}_i^T(p,q,r) = B_i^T(p,q,r)$.*

Proof. Since $e_{i,\xi_0}, l_{i,\xi_0^-} > 0$ and $(p_{\xi_0}, q_{\xi_0}) \neq (0,0)$, we always have $p_{\xi_0} e_{i,\xi_0} + p_{\xi_0} F_{i,\xi_0}(l_{i,\xi_0^-}) + q_{\xi_0} l_{i,\xi_0^-} > 0$. Therefore, we can choose $(c_{i,\xi_0}, l_{i,\xi_0}, b_{i,\xi_0})$ strictly positive, and then $(c_{i,\xi}, l_{i,\xi}, b_{i,\xi})$ strictly positive such that this plan belongs to $C_i^T(p,q,r)$. Note that $p_\xi e_{i,\xi} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + q_\xi l_{i,\xi^-} + r_\xi b_{i,\xi^-} > 0$ if $e_{i,\xi} > 0, l_{i,\xi^-} > 0, b_{i,\xi^-} > 0$ and $(p_\xi, q_\xi, r_\xi) \neq (0,0,0)$. □

Lemma 2. *$C_i^T(p,q,r)$ is lower semi-continuous correspondence on $\mathcal{P}$. $B_i^T(p,q,r)$ is continuous on $\mathcal{P}$ with compact convex values.*

Proof. It is clear since $C_i^T(p,q,r)$ is nonempty and has open graph. □

We now define correspondences.

First, we define φ_0 (for additional agent 0) : $\mathcal{X} \to 2^{\mathcal{P}}$ by

$$\begin{aligned}
\varphi_0((z_i)_{i=1}^m) := \operatorname*{arg\,max}_{(p,q,r)\in\mathcal{P}} \Big\{ & \sum_{\xi \in D^T} \Big[p_\xi \sum_{i=1}^m \big(c_{i,\xi} - e_{i,\xi} - F_{i,\xi}(l_{i,\xi^-})\big) \Big] \\
& + \sum_{\xi \in D^{T-1}} \Big[q_\xi \sum_{i=1}^m (l_{i,\xi} - l_{i,\xi^-}) \Big] + \sum_{\xi \in D^T} \Big[r_\xi (-\sum_{i=1}^m b_{i,\xi^-}) \Big] \Big\}.
\end{aligned}$$

Second, for each $i = 1, \ldots, m$, we define $\varphi_i : \mathcal{P} \to 2^{\mathcal{X}}$

$$\varphi_i((p,q,r)) \quad := \quad \operatorname*{arg\,max}_{(c_i,l_i,b_i)\in C_i^T(p,q,r)} \Big\{ \sum_{t=0}^T u_{i,\xi}(c_{i,\xi}) \Big\}.$$

Lemma 3. *The correspondence φ_i is upper semi-continuous and non-empty, convex, compact valued for each $i = 0, 1, \ldots, m+1$.*

Proof. This is a direct consequence of the Maximum Theorem. □

According to the Kakutani Theorem, there exists $(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m)$ such that

$$(\bar{p}, \bar{q}, \bar{r}) \in \varphi_0((\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m) \tag{17}$$
$$(\bar{c}_i, \bar{l}_i, \bar{b}_i) \in \varphi_i((\bar{p}, \bar{q}, \bar{r})). \tag{18}$$

Denote, for each $\xi \geq \xi_0$,

$$\bar{X}_\xi := \sum_{i=1}^m (\bar{c}_{i,\xi} - e_{i,\xi} - F_{i,\xi}(\bar{l}_{i,\xi^-})), \quad \bar{Y}_\xi := \sum_{i=1}^m (\bar{l}_{i,\xi} - \bar{l}_{i,\xi^-}), \quad \bar{Z}_\xi := -\sum_{i=1}^m \bar{b}_{i,\xi}.$$

For every $(p, q, r) \in \mathcal{P}$, we have

$$\sum_{\xi \in D^T} (p_\xi - \bar{p}_\xi)\bar{X}_\xi + \sum_{\xi \in D^{T-1}} (q_\xi - \bar{q}_\xi)\bar{Y}_\xi + \sum_{\xi \in D^T} (r_\xi - \bar{r}_\xi)\bar{Z}_{\xi^-} \leq 0 \tag{19}$$

Consider node $\xi \in D_T$ at date T, by summing budget constraints over i, we get that

$$\bar{p}_\xi \bar{X}_\xi + \bar{r}_\xi \bar{Z}_{\xi^-} \leq 0.$$

By consequence, we have, for each $\xi \in D_T$ and for every $(p_\xi, r_\xi) \geq 0$ with $2p_\xi + r_\xi = 1$,

$$p_\xi \bar{X}_\xi + r_\xi \bar{Z}_{\xi^-} \leq \bar{p}_\xi \bar{X}_\xi + \bar{r}_\xi \bar{Z}_{\xi^-} \leq 0.$$

Therefore, we have $\bar{X}_\xi \leq$ and $\bar{Z}_{\xi^-} \leq 0$ for each $\xi \in D_T$, which means that

$$\sum_{i=1}^m \bar{c}_{i,\xi} \leq \sum_{i=1}^m (e_{i,\xi} + F_{i,\xi}(\bar{l}_{i,\xi^-})) \text{ and } -\sum_{i=1}^m \bar{b}_{i,\xi^-} \leq 0. \tag{20}$$

Consider a node ξ, by summing the budget constraints over i at this node, we obtain that

$$\bar{p}_\xi \bar{X}_\xi + \bar{q}_\xi \bar{Y}_\xi + \bar{r}_\xi \bar{Z}_{\xi^-} \leq \bar{Z}_\xi \ \forall \xi.$$

Since at date T, we have $\bar{Z}_\xi = 0$ and $\bar{Z}_{\xi^-} \leq 0$ for any $\xi \in D^T$, we get that

$$\bar{p}_\xi \bar{X}_\xi + \bar{q}_\xi \bar{Y}_\xi + \bar{r}_\xi \bar{Z}_{\xi^-} \leq 0 \ \forall \xi \in D_{T-1}.$$

This implies that $\bar{X}_\xi, \bar{Y}_\xi, \bar{Z}_{\xi^-} \leq 0 \ \forall \xi \in D_{T-1}$. Repeating this argument, we obtain that $\bar{X}_\xi, \bar{Y}_\xi, \bar{Z}_\xi \leq 0 \ \forall \xi \in D^T$ which means that

$$\sum_{i=1}^m \bar{c}_{i,\xi} \leq \sum_{i=1}^m (e_{i,\xi} + F_{i,t}(\bar{l}_{i,\xi^-}))$$
$$\sum_{i=1}^m (\bar{l}_{i,\xi} - \bar{l}_{i,\xi^-}) \leq 0$$
$$-\sum_{i=1}^m \bar{b}_{i,\xi} \leq 0.$$

Lemma 4. $\bar{p}_\xi > 0, \bar{r}_\xi > 0$ *for any* $\xi \in D^T$ *and* $\bar{q}_\xi > 0$ *for any* $\xi \in D^{T-1}$.

Proof. We see that $\sum\limits_{i=1}^{m} \bar{l}_{i,\xi} \leq L$ for any ξ, hence

$$\sum_{i=1}^{m} \bar{c}_{i,\xi} \leq \sum_{i=1}^{m}(e_{i,\xi} + F_{i,\xi}(L))$$

which allows us to prove that $\bar{p}_\xi > 0$ for any ξ. Indeed, if $\bar{p}_\xi = 0$ then $c_{i,\xi} = B_c > \sum\limits_{i=1}^{m}(e_{i,\xi} + F_{i,\xi}(L))$, a contradiction.

If $\bar{q}_\xi = 0$, then $\bar{l}_{i,\xi} = B_l > L$ for any i, contradiction. Therefore, $\bar{q}_\xi > 0$

If $\bar{r}_\xi = 0$, then $\bar{b}_{i,\xi^-} = -B_a$ for any i, which implies that $\sum\limits_{i=1}^{m} \bar{b}_{i,\xi-} < 0$, contradiction. Therefore, $\bar{r}_\xi > 0$.

□

Lemma 5. $\bar{X}_\xi = \bar{Y}_\xi = \bar{Z}_\xi = 0$.

Proof. Using $\bar{p}_\xi \bar{X}_\xi + \bar{q}_\xi \bar{Y}_\xi + \bar{r}_\xi \bar{Z}_{\xi^-} \leq 0$ and Lemma 4. Notice that in period T, we only require $\sum_i (l_{i,\xi} - l_{i,\xi^-}) \leq 0$ but $\sum_i (l_{i,\xi} - l_{i,\xi^-}) q_\xi = 0$ as we impose $q_\xi = 0$ for any $\xi \in D_T$. □

Lemma 6. *The optimality of* $(\bar{c}_i, \bar{l}_i, \bar{b}_i)$.

Proof. It is clear since $(\bar{c}_i, \bar{l}_i, \bar{b}_i) \in \varphi_i((\bar{p}, \bar{q}, \bar{r}))$. □

We have just proved that $(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m)$ is an equilibrium for the economy $\tilde{\mathcal{E}}_b^T$. □

We see that any equilibrium of the economy $\tilde{\mathcal{E}}^T$, if it exists, we can normalize by setting $2p_\xi + q_\xi + r_\xi = 1$. We also observe that consumption, land and asset allocations of such an equilibrium are in $\mathcal{X}$, defined in (11). Hence, we obtain that

Proposition 3. *An equilibrium* $(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m)$, *with* $2p_\xi + q_\xi + r_\xi = 1$, *of* $\tilde{\mathcal{E}}_b^T$ *is an equilibrium for* $\tilde{\mathcal{E}}^T$.

2.2 Existence of equilibrium for the infinite-horizon economy

Let us denote $W_\xi := \sum\limits_{i=1}^{m} \big(e_{i,\xi} + F_{i,\xi}(L)\big)$. We need the following assumption to ensure that the utility of every agent is finite.

Assumption 6. $\sum\limits_{\xi \in \mathcal{D}} u_{i,\xi}(W_\xi) < \infty$ *for each* i.

Remark 2. *For simplicity of notation, in what follows, we write* F_i *instead of* $F_{i,\xi}$.

Proposition 4. *Under Assumptions (1-4) and 7, there exists an equilibrium for the economy* $\tilde{\mathcal{E}}$.

Proof. We present a proof in the spirit Le Van and Pham (2016).

We have shown that there exists an equilibrium, say $\big(\bar{p}^T, \bar{q}^T, \bar{r}^T, (\bar{c}_i^T, \bar{l}_i^T, \bar{b}_i^T)_i\big)$, for each $T-$horizon truncated economy $\tilde{\mathcal{E}}^T$. Recall that $2\bar{p}_\xi^T + \bar{q}_\xi^T + \bar{r}_\xi^T = 1$ for any $\xi \in D^T$.

It is clear that, for any $\xi \in D^T$, $0 < \bar{c}_{i,\xi}^T < W_\xi$, $\bar{l}_{i,\xi}^T \in [0, L]$, $\bar{p}_\xi^T \in [0,1], \bar{q}_\xi^T \in [0,1], \bar{r}_\xi^T \in [0,1]$.

We define a sequence (B_ξ) as by $B_{\xi_0} = W_{\xi_0}, B_{\xi'} = B_\xi + W_{\xi'}\ \forall \xi' \in \xi^+$. It is easy to prove that $\bar{b}^T_{i,\xi} < B_\xi$ for any ξ and any T. Since $\sum_{i=1}^m \bar{b}^T_{i,\xi} = 0$ we have $\bar{b}^T_{i,\xi} \in (-mB_\xi, mB_\xi)$ for any ξ and any T. Since $\mathcal{D}$ is a countable set, we can assume that, without loss of generality,

$$\big(\bar{p}^T, \bar{q}^T, \bar{r}^T, (\bar{c}^T_i, \bar{l}^T_i, \bar{b}^T_i)_i)\big) \xrightarrow{T\to\infty} \big(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_i\big)$$

for the product topology.

We will prove that $\big(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_i\big)$ is an equilibrium for the economy $\tilde{\mathcal{E}}$. The market clearing conditions are trivial. We will prove that all prices are strictly positive and the allocation $(\bar{c}_i, \bar{l}_i, \bar{b}_i)$ is optimal.

Let (c_i, l_i, b_i) be a feasible allocation of the problem $P_i(\bar{p}, \bar{q}, \bar{r})$. First, we have to prove that $U_i(c_i) \le U_i(\bar{c}_i)$. Let define $(c'_{i,\xi}, l'_{i,\xi}, b'_{i,\xi})_{\xi\in D^T}$ as follows:

$$(c'_{i,\xi}, l'_{i,\xi}, b'_{i,\xi}) = (c_{i,\xi}, l_{i,\xi}, b_{i,\xi}) \text{ if } \xi \in D^{T-1}$$
$$(c'_{i,\xi}, l'_{i,\xi}, b'_{i,\xi}) = (e_{i,\xi}, 0, 0) \text{ if } \xi \in D_T.$$

We see that $(c'_{i,\xi}, l'_{i,\xi}, b'_{i,\xi})_{\xi\in D^T}$ belongs to $B^T_i(\bar{p}, \bar{q}, \bar{r})$.[3]

Since $l_{i,\xi_0^-} > 0$ and $(\bar{p}_{\xi_0}, \bar{q}_{\xi_0}) \neq (0,0)$, we have $\bar{q}_{\xi_0} l_{i,\xi_0^-} + \bar{p}_{\xi_0} F_i(l_{i,\xi_0^-}) > 0$, and hence $C^T_i(\bar{p}, \bar{q}, \bar{r}) \neq \emptyset$, therefore there exists a sequence $\Big((c^n_{i,\xi}, l^n_{i,\xi}, b^n_{i,\xi})_{\xi\in D^T}\Big)_{n=0}^\infty \in C^T_i(\bar{p}, \bar{q}, \bar{r})$, with $l^n_{i,\xi} = 0$, $b^n_{i,\xi} = 0$ for any $\xi \in D_T$, and this sequence converges to $(c'_{i,\xi}, l'_{i,\xi}, b'_{i,\xi})_{\xi\in D^T}$ when n tends to infinity. We have, for each $\xi \in D^T$,

$$\bar{p}_\xi c^n_{i,\xi} + \bar{q}_\xi l^n_{i,\xi} + b^n_{i,\xi} < \bar{p}_\xi e_{i,\xi} + \bar{q}_\xi l^n_{i,\xi^-} + \bar{p}_\xi F_i(l^n_{i,\xi^-}) + \bar{r}_\xi b^n_{i,\xi^-}$$
$$f_i\Big(\bar{q}_\xi l^n_{i,\xi^-} + \bar{p}_\xi F_i(l^n_{i,\xi^-})\Big) + \bar{r}_\xi b^n_{i,\xi^-} > 0.$$

We can chose s_0 high enough such that (i) $s_0 > T$ and (ii) for every $s \ge s_0$, we have

$$\bar{p}^s_\xi c^n_{i,\xi} + \bar{q}^s_\xi l^n_{i,\xi} + b^n_{i,\xi} < \bar{p}^s_\xi e_{i,\xi} + \bar{q}^s_\xi l^n_{i,\xi^-} + \bar{p}^s_\xi F_i(l^n_{i,\xi^-}) + \bar{r}^s_\xi b^n_{i,\xi^-}$$
$$f_i\Big(\bar{q}^s_\xi l^n_{i,\xi^-} + \bar{p}^s_\xi F_i(l^n_{i,\xi^-})\Big) + \bar{r}^s_\xi b^n_{i,\xi^-} > 0.$$

Condition (ii) means that $(c^n_{i,\xi}, l^n_{i,\xi}, b^n_{i,\xi})_{\xi\in D^T} \in C^T_i(\bar{p}^s, \bar{q}^s, \bar{r}^s)$. Therefore, we get that

$$\sum_{\xi\in D^T} u_{i,\xi}(c^n_{i,\xi}) \le \sum_{\xi\in D^T} u_{i,\xi}(\bar{c}^s_{i,\xi}).$$

Let s tend to infinity, we obtain $\sum_{\xi\in D^T} u_{i,\xi}(c^n_{i,\xi}) \le \sum_{\xi\in D^T} u_{i,\xi}(\bar{c}_{i,\xi}) \le \sum_{\xi\in\mathcal{D}} u_{i,\xi}(\bar{c}_{i,\xi})$ for any n, T. Let n tend to infinity, we have $\sum_{\xi\in D^T} u_{i,\xi}(c'_{i,\xi}) \le \sum_{\xi\in\mathcal{D}} u_{i,\xi}(\bar{c}_{i,\xi})$ for any T, which implies that

$$\sum_{\xi\in D^{T-1}} u_{i,\xi}(c_{i,\xi}) \le \sum_{\xi\in D_T} u_{i,\xi}(e_{i,\xi}) + \sum_{\xi\in D^{T-1}} u_{i,\xi}(c_{i,\xi}) \le \sum_{\xi\in\mathcal{D}} u_{i,\xi}(\bar{c}_{i,\xi}). \tag{21}$$

Let T tend to infinity in (21), we get that

$$\sum_{\xi\in\mathcal{D}} u_{i,\xi}(c_{i,\xi}) \le \sum_{\xi\in\mathcal{D}} u_{i,\xi}(\bar{c}_{i,\xi}).$$

[3] Thank to borrowing constraints and $f_i \le 1$, we can choose $c'_{i,\xi} = e_{i,\xi}$ for any $\xi \in D_T$.

So, we have proved the optimality of $(\bar{c}_i, \bar{l}_i, \bar{b}_i)$.

Now, we prove that $\bar{p}_\xi > 0$. Indeed, if $\bar{p}_\xi = 0$, the agent i can freely improve her consumption to obtain a level of utility, which is higher than $\sum_{\xi \in \mathcal{D}} u_{i,\xi}(\bar{c}_{i,\xi})$. This contradicts the optimality of $(\bar{c}_i, \bar{l}_i, \bar{b}_i)$.

We have $\bar{q}_\xi$ is strictly positive because otherwise we can choose another allocation such that $l'_{i,\xi} = \infty$ and at the next date, $c'_{i,\xi'} = \infty \ \forall \xi' \in \xi^+$, which make the utility of agent i infinity, contradiction. By using a similar argument, we can prove $\bar{r}_\xi > 0$.

□

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