Sustainability Narrowness

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Abstract

We study the resilience of a multiplex socio-ecological system (SES) which we structure from the spheres composing the sustainability Venn diagram. The SES network is subject to dynamics of spread of a global reform through the knock-on effect. The model outcomes reveal that high probability of reform completion on an SES layer through nodes previously reformed on other SES layers is necessary and sufficient to obtain positive density of reformed nodes on that layer. Full density can only be reached in the absence of risk of reform abrogation. The opposite case prevents the equilibrium density from reaching a steady state. The numerical simulation results show that the combination of likely probability of reform completion and of proportional influence of all layers yields the maximum magnitude of efficiency of the knock-on effect. We thus provide a formalized argument in favor of giving equal weight to all aspects of sustainable development.

Keywords: bioeconomics, socio-ecological systems, multiplex networks, sustainability, resilience

1 Introduction

The concepts of sustainable development and sustainability have been introduced to reduce economic disparities, social exclusion and environmental degradation. No need to remind that the concept of sustainability came to the fore after the publication of the well-known Brundtland commission report (Tomlinson, 1987). Among different representations of sustainability, the Venn diagram (Fig. 1), where the three spheres of economy, society and environment overlap (Mebratu, 1998), is the most widespread and referred to.[1] The representation involves the simultaneous pursuit of economic prosperity, social equity and environmental quality (Elkington, 2002). As can be easily noticed, the area in which all the objectives coincide is very narrow. For instance, in a study conducted by Estapé-Dubreuil et al. (2016), the authors do not find a decision-making factor used in micro-investing that takes all three dimensions of sustainability into account.

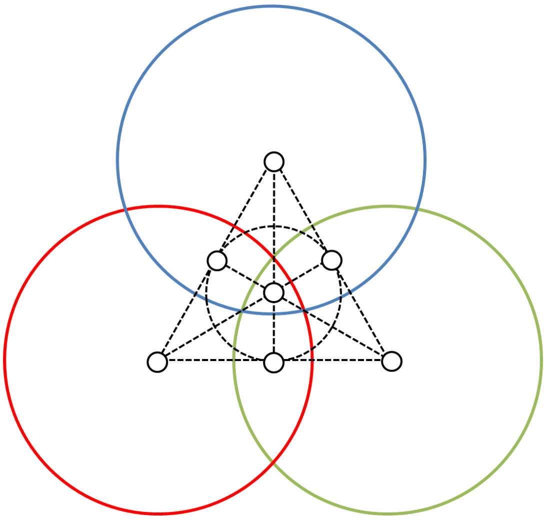

Figure 1: Graphical representation of sustainability in form of a Venn diagram composed of economic (blue), social (red) and environmental (green) areas. The inside network corresponds to the Fano plane.

In the outer years, frameworks laying bare the interconnectedness of humans and their environment, such as socio-ecological systems (SES), have been developed (Ostrom, 2009). They are considered to better describe the dynamics of interactions between human communities and their environment (Waltner-Toews et al., 2008). Indeed, polycentric

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[1] Despite the alternative representation in form of concentric circles (Mitchell, 2000), where the economic area is embedded in the social area, itself being inside the natural environment, the reasoning behind seems rather distant from the current mentalities.

systems are better adapted to social-ecological dynamics because these coupled systems effectively link scales via diverse information flow capabilities (Ostrom, 2010). In detail, SES are composed of anthropogenic and natural elements interacting through temporal, spatial and organizational scales. When SES are represented in form of a network, the latter is composed of nodes, such as natural components, resource users, civil players, voters, economic actors or regulatory organizations, and of linkages between those nodes, like exchanges or transfers of money, energy, information and strategies.

The analysis of SES sustainability has been principally conducted through that of resilience, provided that collapses result from the lack of resilience (Gonzalès and Parrott, 2012).[2] Therefore, Carpenter et al. (2012) emphasized that public policies for general resilience must overcome budget limitations, address trade-offs, be acceptable to many competing interests, and overcome barriers in the structure of existing institutions. Managing for resilience then requires legal framework to be reformed in order to accommodate the SES dynamic processes (Garmestan and Benson, 2013). As a matter of fact, adaptive management is unlikely to be effective without reform, and without adaptive management, environmental governance is unlikely to succeed (Ruhl, 2005). In continuity of their insights, we wish to study the possibility of reform on multilayered networks as well as to measure the efficiency of spread of change in such systems, should their actual state be reputed to be obsolete or in jeopardy.[3]

The exemplification of the foregoing can be done through the recommendation for fiscal reforms in G20 countries, that would at once benefit economic growth, social inclusion and environmental outcomes, which entails significant changes in tax structures, increased emphasis on environmental taxation and a review of environmentally harmful subsidies (OECD et al., 2012). Another example within reach is relative to the Sendai treaty for disaster risk reduction (Aitsi-Selmi et al., 2015) as a major agreement of the post-2015 development agenda; the latter is voluntary and non-binding, in which the stakeholders issued from different backgrounds need to make a concerted effort at reducing the impact of a natural hazard. But how can a decentralized change of such scope be implemented? To answer that question, we consider sustainability to be achievable, by means of a reform initiated by public authorities or the civil society, whenever the states of SES components

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[2] A system is considered to be resilient when its structure adapts to perturbations while continuing to function, be it at the expense of changes (Liu et al., 2007).

[3] The concept of reform assumes the presence of a crisis which could be solved through corrective actions.

or nodes can be efficiently updated. Put differently, if the update among the system constituents spreads sufficiently, the system is considered to be controllable, in the sense that some agents manage to drive the others,[4] via the network connectivity, toward the objectives at stake. In consequence, allowing for the overall system controllability, through the medium of control theory, becomes a necessary condition to ensure sustainability.[5]

The Fano plane (Fig. 1) corresponds to the network variant of the Venn diagram. If we concieve sustainability as achievable through the spread of reform among all constituents, it would necessitate that all Laplacian eigenvalues – which denote the number of connected components in a graph – be distinct, which cannot be verified in a triple Steiner system (Aguilar and Gharesifard, 2015). This implies that the system controllability, for the purpose of sustainability attainment, is theoretically impossible to achieve. On the other side, interconnected multiplexes or multilayered networks are a class of dynamic networks introduced to model real-world complex systems, in which the nodes are connected via more than one type of links (Mucha et al., 2010; Lee et al., 2015). The particularity of a multiplex system is the functional coupling between the layers of a certain kind. The network layers are then constituted of links of different types. The field has become one of the major contemporary topics in network theory (Lee et al., 2015).

In order to go beyond the limits imposed by the Fano plane, we decide to study resilience through a diffusion of a global reform on a multiplex SES network. The latter can be envisaged as such since SES express their robustness through the ability to change over time (Mucha et al., 2010; Gonzalès and Parrott, 2012). That way, we structure the network from the three spheres forming the Venn diagram, with a functional coupling between economy, society and environment, which we subject to dynamics of reform through the knock-on effect, such that the spread of reform on a node comes from the neighborhood or from the counterparts previously reformed. This approach is motivated by the lack of dynamic perspectives and of full interrelatedness among the components of sustainability (Lozano, 2008b).

The model outcomes reveal that high probability of reform completion on an SES layer through nodes previously reformed on other SES layers is necessary and sufficient to obtain positive density of reformed nodes on that layer. As for full density, it can only be reached

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[4] The leading of agents concerns the practice of negotiations and discussions.

[5] Shastri et al. (2008) use a system theory-based approach, with an optimal control problem formulation, in the interest of deriving time-dependent management strategies.

in the absence of risk of reform abrogation. The opposite case prevents the equilibrium density from reaching a steady state. The numerical simulation results show that the combination of likely probability of reform completion and of proportional influence of all layers yields the maximum magnitude of efficiency of the knock-on effect. We thus provide a formalized argument in favor of giving equal weight to all aspects of sustainable development. Our clarification also opens an interesting debate on sustainability issues.

After this starting section, the dynamic behavior of the multiplex network, studied at the levels of a layer and of multiple layers, is modeled in Section 2. Section 3 is devoted to illustrating simulation examples. Section 4 discusses the implications of the theoretical results.

2 Model

Following the methodology developed by Wei et al. (2016), we consider an interacting multiplex network, such as the one depicted in Fig. 2, in which the population of agents[6] is distributed among L_n layers, where $n = 1, ..., 3$. Each layer contains N nodes, with $i = 1, ..., N$, with different intra-layer connectivity. Let A^n, for $n = 1, ..., 3$, be the adjacency matrix of L_n with nonnegative elements $(a^n_{ij})_{N \times N}$, for $i = 1, ..., N$. Two nodes are connected when $a^n_{ij} = 1$; and $a^n_{ij} = 0$ otherwise. Each node in L_n is connected to its counterparts in L_{-n}, such that there exists a systematic link between the nodes of different layers.

Agents can either be target nodes or reformed nodes, which, in the latter case, have previously been targets of the reform. In order to get reformed, a target node has to be sufficiently open for reform. In case a target node from L_n is connected to at least one intra-layer reformed node, let $\beta_n \in [0, 1]$, $n = 1, ..., 3$, be the probability of openness for reform on layer n. Should the reform be called into question, let $\mu_n \in [0, 1]$, $n = 1, ..., 3$, be the risk that a reformed node from L_n gets abrogated, such that it returns back to its original state.

Given the inter-layer connectivity of the multiplex network, let k_n, such that $\sum_{n=1}^{3} k_n = 1$, be the parameter of influence emanating from either layer. For example, whenever a

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[6] For illustrative purposes, the population of agents can be interpreted as a set of countries, the actors of which evolve in different SES layers, that are pursuing common reforms, or as a set of stakeholders, representing various spheres of the society, committed to the same purpose.

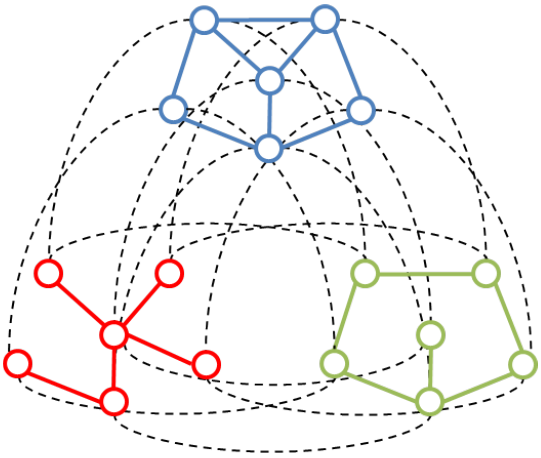

Figure 2: Example of a multiplex network composed of economic (blue), social (red) and environmental (green) layers. Each of them is composed of six connected nodes, that is $L_n = \{1, 2, 3, 4, 5, 6\}$, for $n = 1, ..., 3$.

reform comes from the counterparts in other layers, the weighted probability that a target node from L_n is open for reform issued by other layers amounts to $(1 - k_n)\beta_n$.

Finally, consider $p_{n,i}(t) \in [0, 1]$ to be the probability that node i from L_n gets reformed at time t, such that its complement corresponds to the probability that i remains a target node.[7]

2.1 Intra-layer connectivity

2.1.1 Dynamics

Before moving forward to multiplex networks, let us start with a single layer in order to study the intra-layer connectivity. The evolution of p for node i from L_n is formalized in the form of a dynamical equation

$$\begin{aligned} p_{n,i}(t+1) &= (1 - p_{n,i}(t))(1 - q_{n,i}(t)) + p_{n,i}(t)(1 - \mu_n) \\ &= 1 - q_{n,i}(t) - p_{n,i}(t) + p_{n,i}(t)q_{n,i}(t) + p_{n,i}(t) - p_{n,i}(t)\mu_n \\ &= 1 + p_{n,i}(t)(q_{n,i}(t) - \mu_n) - q_{n,i}(t) \qquad (1) \end{aligned}$$

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[7] The network model puts emphasis on the fact that agents do not necessarily engage in a binding cooperative game, but instead follow (allow themselves to be influenced by) their neighborhood or (by) their counterparts evolving on other layers. In point of fact, in dynamic game theory, the spread of a strategy, or that of a practice, takes place after the individual comparison between alternative payoffs, where high-payoff strategies propagate in the population of players.

for $n = 1\backslash\{2,3\}$ and $i = 1, ..., N$, where $q_{n,i}(t) = \prod_{j=1}^{N}\left(1 - \beta_n a_{ij}^n p_{n,j}(t)\right)$ represents the probability that node i, despite being open for reform, does not get reformed by neighbor j.

The discrete dynamics corresponds to the sum of the composed probability that open node i is reformed by at least one neighbor and of the composed probability that reformed node i is not subject to abrogation. Rewriting the equation enables us to highlight the fact that the level of $p_{n,i}$ at time step $t+1$ depends on the gap between the probabilities of reform failure and abrogation at time t.

According to the technique used in Dragicevic (2015), solving the dynamical equation reduces to solving the nonhomogeneous recurrence relation $p_{n,i}(t)$, where $c_1(q_{n,i}(t) - \mu_n)$ is the associated homogeneous recurrence relation with a solution of $c_1(q_{n,i}(t) - \mu_n)^{t-1}$. The nonhomogeneous part yields $c_2 = \frac{(q_{n,i}(t)-1)(1-(q_{n,i}(t)-\mu_n)^t)}{(q_{n,i}(t)-\mu_n-1)}$ from which we obtain the stationarity expression of

$$p_{n,i}^{\star}(t) = \frac{(q_{n,i}(t)-1)(1-(q_{n,i}(t)-\mu_n)^t)}{q_{n,i}(t)-\mu_n-1}\left[\frac{(q_{n,i}(t)-\mu_n)^t}{(q_{n,i}(t)-\mu_n)^t-(q_{n,i}(t)-\mu_n)^{t-1}}\right] \tag{2}$$

for $n = 1\backslash\{2,3\}$ and $i = 1, ..., N$. After considering the above, three cases may be observed.

The first case corresponds to $p_{n,i}^{\star}(t) = 0 \Leftrightarrow q_{n,i}(t) = \mu_n$. The probability that node $i \in L_n$ gets reformed at time t by any neighbor from the layer is null if the probability that a target node does not get reformed is equal to the probability that a reformed node becomes abrogated.

The second case corresponds to $p_{n,i}^{\star}(t) > 0 \Leftrightarrow q_{n,i}(t) < \mu_n$. The probability that node $i \in L_n$ gets reformed at time t by any neighbor from the layer is positive if the probability that a target node does not get reformed is less than the probability that a reformed node becomes abrogated.

The third case corresponds to $p_{n,i}^{\star}(t) = 1 \Leftrightarrow \mu_n = 0$. The certainty that node $i \in L_n$ gets reformed at time t by any neighbor from the layer occurs if the probability that a reformed node becomes abrogated is equal to zero. The following proposition ensues.

Proposition 1 *In a network exclusively dependent on intra-layer connectivity, although the risk of abrogation might annul a reform project conducted on that layer, a higher level* *than that of reform failure is necessary and sufficient to achieve the possibility of reform; the certainty of reform implies the absence of risk of abrogation.*

The necessity is straightforward from the expression of $p^{\star}_{n,i}(t)$. The sufficiency comes from the construction of probability $q_{n,i}(t)$, which is itself dependent on $p_{n,i}(t)$.

2.1.2 Density

The probability dynamics previously obtained enable us to study the density of reformed agents in a network exclusively dependent on intra-connectivity.

$$\begin{aligned}\rho^{\star}_{i}(t) &= \frac{1}{N}\sum_{j=1}^{N} p^{\star}_{n,i}(t) \\ &= \frac{1}{N}\sum_{j=1}^{N} \frac{(q_{n,i}(t)-1)(1-(q_{n,i}(t)-\mu_n)^t)}{q_{n,i}(t)-\mu_n-1}\left[\frac{(q_{n,i}(t)-\mu_n)^t}{(q_{n,i}(t)-\mu_n)^t-(q_{n,i}(t)-\mu_n)^{t-1}}\right]\end{aligned} \tag{3}$$

for $n = 1\backslash\{2,3\}$ and $i = 1,...,N$, where $q_{n,i}(t) = \prod_{j=1}^{N}\left(1-\beta_n a^n_{ij} p^{\star}_{n,j}(t)\right)$ is the probability that node i, despite being open for reform, does not get reformed by neighbor j.

We observe that $\rho^{\star}_{i}(t) = 0 \Leftrightarrow q_{n,i}(t) = \{1, \mu_n\}$. This implies that the density of reformed nodes is equal to zero in case of certainty that node i has not been reformed by any neighbor from L_n at time t, or when its probability of not being reformed equals that of being abrogated. In greater depth, these properties give the following.

$$\begin{aligned} q_{n,i}(t) = 1 &\Leftrightarrow \prod_{j=1}^{N}\left(1-\beta_n a^n_{ij} p^{\star}_{n,j}(t)\right) = 1 \\ &\Leftrightarrow \left(1-\beta_n a^n_{ij} p^{\star}_{n,j}(t)\right)^N = 1 \\ &\Leftrightarrow \beta_n = \frac{1}{a^n_{ij} p^{\star}_{n,j}(t)} = 0 \end{aligned} \tag{4}$$

for $n = 1\backslash\{2,3\}$ and $i = 1,...,N$. While $a^n_{ij}p^{\star}_{n,j}(t) = 1/\beta_n$ represents the eigenvalue of the adjacency matrix A, the reversed expression, that is $\beta_n = 1/a^n_{ij}p^{\star}_{n,j}(t)$, corresponds to the spillover threshold for the policy engaged in L_n. In this case, it amounts to zero, which implies that the policy knock-on effect shall be sterile. The second equality yields

$$
\begin{aligned}
q_{n,i}(t) = \mu_n \quad &\Leftrightarrow \quad \prod_{j=1}^{N} \left(1 - \beta_n a_{ij}^{n} p_{n,j}^{\star}(t)\right) = \mu_n \\
&\Leftrightarrow \quad \left(1 - \beta_n a_{ij}^{n} p_{n,j}^{\star}(t)\right)^{N} = \mu_n \\
&\Leftrightarrow \quad \beta_n = \frac{1 - \mu_n^{\frac{1}{N}}}{a_{ij}^{n} p_{n,j}^{\star}(t)} \qquad (5)
\end{aligned}
$$

for $n = 1\backslash\{2,3\}$ and $i = 1, ..., N$, where the spillover threshold amounts to $\frac{1-\mu_n^{1/N}}{a_{ij}^{n} p_{n,j}^{\star}(t)}$. We have $\lim_{p_{n,j}^{\star}(t) \to 0} \frac{1-\mu_n^{1/N}}{a_{ij}^{n} p_{n,j}^{\star}(t)} = \infty$ and $\lim_{p_{n,j}^{\star}(t) \to 1} \frac{1-\mu_n^{1/N}}{a_{ij}^{n} p_{n,j}^{\star}(t)} = 1 - \mu_n^{1/N}$. In the first case, as the probability of reforming node i by neighbor j at time t goes to zero, the spillover threshold goes to the unattainable level of infinity, which yields a zero density of reformed nodes. In the second case, as the probability of reforming node i by neighbor j at time t tends to one, the spillover threshold amounts to $1 - \mu_n^{1/N}$, that is zero for large values of N. We thus fall on the same property in both cases.

Likewise, we observe that $\rho_i^{\star}(t) > 0 \Leftrightarrow q_{n,i}(t) < \mu_n$. The result implies that the density of reformed nodes is strictly positive when the probability that node i has not been reformed by any neighbor at time t is less than that of being abrogated. In detail, we have the following

$$
\begin{aligned}
q_{n,i}(t) < \mu_n \quad &\Leftrightarrow \quad \prod_{j=1}^{N} \left(1 - \beta_n a_{ij}^{n} p_{n,j}^{\star}(t)\right) < \mu_n \\
&\Leftrightarrow \quad \left(1 - \beta_n a_{ij}^{n} p_{n,j}^{\star}(t)\right)^{N} < \mu_n \\
&\Leftrightarrow \quad \frac{1 - \mu_n^{\frac{1}{N}}}{a_{ij}^{n} p_{n,j}^{\star}(t)} < \beta_n < \frac{1}{a_{ij}^{n} p_{n,j}^{\star}(t)} \qquad (6)
\end{aligned}
$$

for $n = 1\backslash\{2,3\}$ and $i = 1, ..., N$. We have $\lim_{p_{n,j}^{\star}(t) \to 0} \left\{ \frac{1-\mu_n^{1/N}}{a_{ij}^{n} p_{n,j}^{\star}(t)}, \frac{1}{a_{ij}^{n} p_{n,j}^{\star}(t)} \right\} = (\infty, \infty)$ and $\lim_{p_{n,j}^{\star}(t) \to 1} \left\{ \frac{1-\mu_n^{1/N}}{a_{ij}^{n} p_{n,j}^{\star}(t)}, \frac{1}{a_{ij}^{n} p_{n,j}^{\star}(t)} \right\} = \left(1 - \mu_n^{1/N}, 1\right)$. When the probability of reforming node i by neighbor j at time t goes to zero, the interval in which stands the spillover threshold tends to the unrealistic level of infinity. On the contrary, as the probability of reforming node i by neighbor j at time t approaches certainty, the spillover threshold lies within zero and one for large values of N. Therefore, a substantial high probability of reform completion enables to reach positive density of reformed nodes.

Finally, $\rho_i^\star(t) = 1 \Leftrightarrow \mu_n = 0$. In consequence, the absence of risk of reform abrogation enables to strike full density of reformed nodes.

Proposition 2 *In a network exclusively dependent on intra-layer connectivity, high probability of reform completion is necessary and sufficient to obtain positive density of reformed nodes on that layer; full density can only be reached in the absence of risk of reform abrogation.*

2.2 Intra- and inter-layer connectivities

The consideration of an interacting multiplex network makes the density both dependent on intra- and inter-layer connectivities. We have

$$\begin{aligned}\rho_i^\star(t) &= \frac{1}{N}\sum_{j=1}^{N} p_{n,i}^\star(t) \\ &= \frac{1}{N}\sum_{j=1}^{N} \frac{(q_{n,i}(t)-1)(1-(q_{n,i}(t)-\mu_n)^t)}{q_{n,i}(t)-\mu_n-1}\left[\frac{(q_{n,i}(t)-\mu_n)^t}{(q_{n,i}(t)-\mu_n)^t-(q_{n,i}(t)-\mu_n)^{t-1}}\right] \end{aligned} \tag{7}$$

for $n = 1, ..., 3$ and $i = 1, ..., N$, where $q_{n,i}(t) = \prod_{j=1}^{N}\left(1 - k_{-n}\beta_n a_{ij}^n p_{n,j}^\star(t)\right)$ is the probability that node i, despite being open for reform, does not get reformed by neighbor j either through intra- or inter-layer connectivity. This time, $q_{n,i}(t)$ is also dependent on k_{-n}, be it the influence coming from the reformed counterparts in other layers.

Once again, we observe that $\rho_i^\star(t) = 0 \Leftrightarrow q_{n,i}(t) = \{1, \mu_n\}$. More specifically, $k_{-n}\beta_n = (1-k_n)\beta_n = \frac{1}{a_{ij}^n p_{n,j}^\star(t)} = 0$. Despite the influence of nodes from both the neighborhood of node i and from layers L_{-n} through k_{-n}, the policy knock-on effect will be vain. When $q_{n,i}(t) = \mu_n$, the spillover threshold amounts to $\beta_n = \frac{1-\mu_n^{1/N}}{k_{-n}a_{ij}^n p_{n,j}^\star(t)} = \frac{1-\mu_n^{1/N}}{(1-k_n)a_{ij}^n p_{n,j}^\star(t)}$. We have $\lim_{k_n \to 0} \frac{1-\mu_n^{1/N}}{(1-k_n)a_{ij}^n p_{n,j}^\star(t)} = \frac{1-\mu_n^{1/N}}{a_{ij}^n p_{n,j}^\star(t)}$. By that, when the combined influence from layers L_{-n} is high enough, their policy knock-on effect will depend on the probability that node i gets reformed by node j via k_{-n} at time t. As $p_{n,j}^\star(t) \to 1$, the spillover threshold is zero for large values of N. In parallel, we have $\lim_{k_n \to 1} \frac{1-\mu_n^{1/N}}{(1-k_n)a_{ij}^n p_{n,j}^\star(t)} = \infty$, be it another unattainable threshold level. In both cases, zero density of reformed nodes will be achieved.

Again, we observe that $\rho_i^\star(t) > 0 \Leftrightarrow q_{n,i}(t) < \mu_n$. This comes down to $\frac{1-\mu_n^{1/N}}{(1-k_n)a_{ij}^n p_{n,j}^\star(t)} < \beta_n < \frac{1}{(1-k_n)a_{ij}^n p_{n,j}^\star(t)}$ or $\beta_n \in (0, 1)$, when $p_{n,j}^\star(t) \to 1$, for large values of N. When the likelihood of reforming node i by node j through k_{-n} at time t is close to certainty, the spillover threshold lies within zero and one for large values of N. Thereby, $\rho_i^\star(t) > 0$ can be obtained through high probability of achieving reform in other layers.

As for $\rho_i^\star(t) = 1 \Leftrightarrow \mu_n = 0$, reaching full density of reformed nodes implies a risk of reform abrogation equal to zero.

Proposition 3 *In an interacting multiplex network both dependent on intra and inter-layer connectivities, high probability of reform completion on a layer through nodes reformed on other layers is necessary and sufficient to obtain positive density of reformed nodes on that layer; full density can only be reached in the absence of risk of reform abrogation.*

Let us now analyze the stability of equilibrium density by considering $\rho_i^\star(t)$ as a Lyapunov function candidate. The latter is then assumed to be a rate function (Mesquita and Hespanha, 2010). The time derivative is equal to

$$
\begin{aligned}
\rho_i'^{\star}(t) \quad &= \quad \frac{(q_{n,i}(t)-1)(1-(q_{n,i}(t)-\mu_n)^t)}{q_{n,i}(t)-\mu_n-1}\left(\frac{q_{n,i}(t)-\mu_n}{(q_{n,i}(t)-\mu_n)^t-(q_{n,i}(t)-\mu_n)^{t-1}}\right)^{N-1} \qquad (8)\\
&\times \quad \left[\frac{q'_{n,i}(t)}{(q_{n,i}(t)-\mu_n)^t-(q_{n,i}(t)-\mu_n)^{t-1}}\right]\\
&- \quad \left(\frac{q_{n,i}(t)-\mu_n}{(q_{n,i}(t)-\mu_n)^t-(q_{n,i}(t)-\mu_n)^{t-1}}\right)^{N-1}\\
&\times \quad \left[\frac{(q_{n,i}(t)-\mu_n)\left(q_{n,i}(t)-\mu_n\right)^t\left(\frac{tq'_{n,i}(t)}{q_{n,i}(t)-\mu_n}+ln(q_{n,i}(t)-\mu_n)\right)}{\left((q_{n,i}(t)-\mu_n)^t-(q_{n,i}(t)-\mu_n)^{t-1}\right)^2}\right]\\
&+ \quad \left(\frac{q_{n,i}(t)-\mu_n}{(q_{n,i}(t)-\mu_n)^t-(q_{n,i}(t)-\mu_n)^{t-1}}\right)^{N-1}\\
&\times \quad \left[\frac{(q_{n,i}(t)-\mu_n)\left(q_{n,i}(t)-\mu_n\right)^{t-1}\left(\frac{(t-1)q'_{n,i}(t)}{q_{n,i}(t)-\mu_n}+ln(q_{n,i}(t)-\mu_n)\right)}{\left((q_{n,i}(t)-\mu_n)^t-(q_{n,i}(t)-\mu_n)^{t-1}\right)^2}\right]\\
&+ \quad \frac{(q_{n,i}(t)-\mu_n)^t}{(q_{n,i}(t)-\mu_n)^t-(q_{n,i}(t)-\mu_n)^{t-1}}\left(\frac{(q_{n,i}(t)-1)(1-(q_{n,i}(t)-\mu_n)^t)}{q_{n,i}(t)-\mu_n-1}\right)^{N-1}\\
&\times \quad \left[\frac{(1-(q_{n,i}(t)-\mu_n)^t)q'_{n,i}(t)}{q_{n,i}(t)-\mu_n-1}-\frac{(q_{n,i}(t)-1)((1-(q_{n,i}(t)-\mu_n)^t)q'_{n,i}(t)}{(q_{n,i}(t)-\mu_n-1)^2}\right]\\
&- \quad \left(\frac{(q_{n,i}(t)-1)(1-(q_{n,i}(t)-\mu_n)^t)}{q_{n,i}(t)-\mu_n-1}\right)^{N-1}\\
&\times \quad \left[\frac{(q_{n,i}(t)-1)(q_{n,i}(t)-\mu_n)^t\left(\frac{tq'_{n,i}(t)}{q_{n,i}(t)-\mu_n}\right)+ln(q_{n,i}(t)-\mu_n)}{q_{n,i}(t)-\mu_n-1}\right]\\
&\gtrless \quad 0
\end{aligned}
$$

We know, by definition of $q_{n,i}(t)$, that its derivative resumes to that of $-p'_{n,j}{}^{\star}(t) < 0$. As a consequence, whenever $q_{n,i}(t) < \mu_n \leq 1$, which corresponds to the criterion for obtaining positive density of reformed nodes, $\rho_i'^{\star}(t) > 0$, such that the equilibrium density is unstable in the sense of Lyapunov. It implies that the reform spread on layers can be withdrawn in time. The result is in accordance with our previous results, for positive density also depends on the tradeoff between the risks of failing to reform and that of abrogating the reform. In fact, according to the model outcomes, high probability of abrogation signifies that the reform has been previously adopted by a number of nodes. Albeit, what triggers the reform diffusion also prevents it from attaining stationarity.

Proposition 4 *In an interacting multiplex network both dependent on intra- and inter-layer connectivities, high risk of reform abrogation prevents the equilibrium density of* *reformed nodes from reaching a steady state.*

3 Simulations

Based on the properties and conditions previously obtained, the aim of this section is to illustrate, through simulations, the levels of spillover thresholds as well as the potential measures of diffusion.

3.1 Intra-layer connectivity

Fig. 3 illustrates the spillover threshold values above which the policy knock-on effect is operational. We observe a sequence of decreasing convex curves with a corner equilibrium, at $\left\{p^{\star}_{n,j}(t), \mu_n\right\} = (0,1)$, from which arise the belt-shaped areas, that delimit the levels of β_n, colored in shades of blue. It verifies the property of $\mu_n = q_{n,i}(t)$. The same can be noticed for $p^{\star}_{n,i}(t) = 1$, where $\mu_n = 0$.

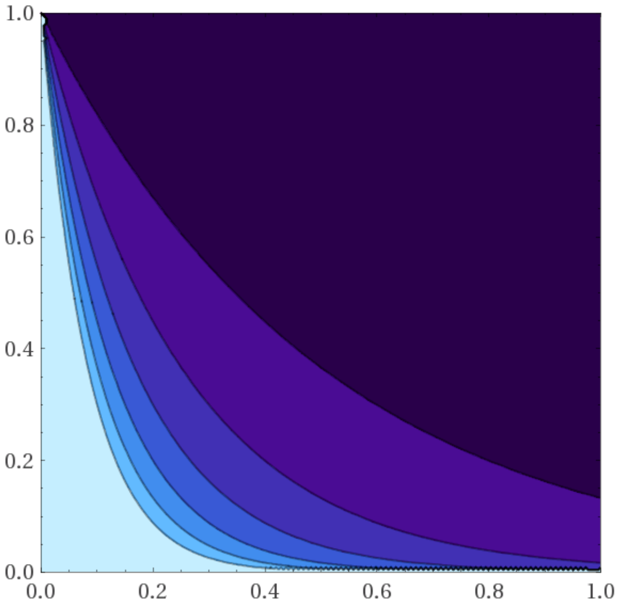

Figure 3: Levels of spillover thresholds β_n in a single layer. The x-axis corresponds to the probability $(p^{\star}_{n,j}(t))$ that node i from L_n is reformed by neighbor j at time t. The y-axis denotes the probability (μ_n) that a reformed node from L_n gets abrogated. While the light blue area corresponds to higher values of spillover threshold, that is $\lim_{p^{\star}_{n,j}(t)\to 0, \mu_n \to 0} \beta_n = 0.23$, dark blue areas match with levels of spillover threshold of $\lim_{p^{\star}_{n,j}(t)\to 1, \mu_n \to 1} \beta_n = 0.00^{+}$.

The substitutability between the probability of reform and that of abrogation is less pronounced for low values of $p^{\star}_{n,i}(t)$ and μ_n. This can be explained by the fact that when the probability of reforming a node is low, the possibility to abrogate that reform is low as well, such that the two parameters evolve in a complementary way. As the eventuality of abrogation increases, the probability of reform decreases, so that both parameters turn substitutable.

For high values of $p^{\star}_{n,j}(t)$, β_n is invariably around zero. A spillover threshold close to but different from zero implies that the reform can easily spread through the intra-layer connectivity. As both $p^{\star}_{n,i}(t)$ and μ_n tend to zero, the spillover threshold increases, and the spread by means of intra-layer connectivity becomes less reachable as well.

Result 1 *In a network exclusively dependent on intra-layer connectivity, likely probability of reform completion irrespective of the probability of reform abrogation is necessary and sufficient to initiate the knock-on effect.*

The first result implies that a decentralized spread of reform on a layer can be conducted through a few nodes only. In consequence, in absence of a central authority which would otherwise impose a vast reform through binding policies, a non-binding directive could be implemented by means of the spillover effect.

3.2 Intra- and inter-layer connectivities

Let us now take a closer look at the combined influence from k_{-n} on the spillover threshold.

Fig. 4 also depicts the values of spillover threshold above which the policy knock-on effect is operational. We observe increasing concave curves, delimiting a series of belt-shaped areas colored in shades of blue, with a gradual transition from complementarity to substitutability. The proportional distribution of knock-on effects coming from layers L_{-n}, where $(1 - k_n) \simeq 2/3$, matches with corner values of $\left\{p^{\star}_{n,j}(t), k_n\right\} = (0^+, 0^+) \cup (1, 1)$.

One interesting result is that β_n only exists for $k_n \leq 1/3$ when $p^{\star}_{n,j}(t) \to 0$. Thereby, whenever the influence from L_{-n} is less than 2/3, at the levels of probabilities of reform – be it through the inter-layer connectivity – close to zero, the knock-on effect fails to function. For $k_n \to 1$, β_n only exists for $p^{\star}_{n,j}(t) \geq 2/3$. In this case, the knock-on effect will not take place either.

For all other configurations, the spread of reform should be operational, with a maximum magnitude of efficiency for $p^{\star}_{n,j}(t) > 1/2$ and $k_n \leq 1/3$.

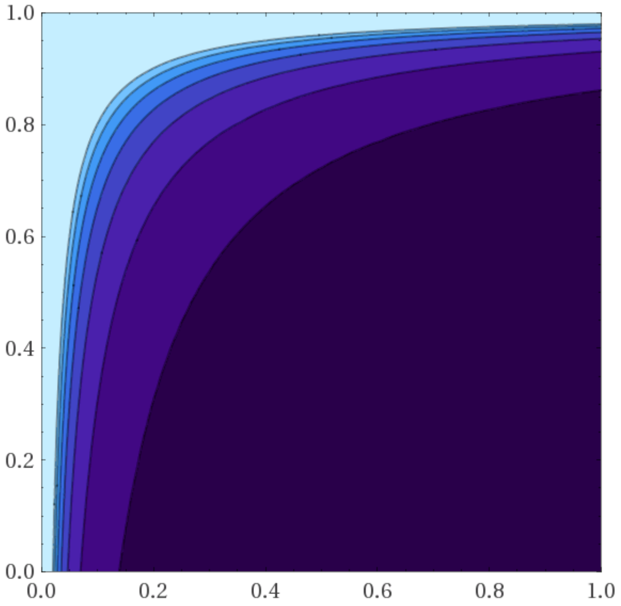

Figure 4: Levels of spillover thresholds β_n in a multiplex network. The x-axis corresponds to the probability ($p^{\star}_{n,j}(t)$) that node i from L_n is reformed by neighbor j through layers L_{-n}. The y-axis denotes the magnitude of influence (k_n) from L_n. While the light blue area corresponds to high levels of spillover threshold, that is $\lim_{p^{\star}_{n,j}(t)\to 0, k_n\to 1} \beta_n = 0.99$, dark blue areas correspond to low but positive levels of spillover threshold, that is $\lim_{p^{\star}_{n,j}(t)\to 1, k_n\to 0} \beta_n = 0.00^{+}$.

Result 2 *In an interacting multiplex network both dependent on intra- and inter-layer connectivities, the combination of likely probability of reform completion and of proportional influence of all layers yields the maximum magnitude of efficiency of the knock-on effect.*

The second result suggests that a decentralized spread of reform can also be conducted through a few nodes only. Nevertheless, unlike the previous case, a non-binding directive, which would this time be addressed to the entire multiplex network, could only be implemented – through the spillover effect – by virtue of a proportional consideration of the counterparts from other layers.

4 Discussion

Many definitions of sustainable development have been proposed, most of which have been previously collected by Kirby et al. (1995). According to Lozano (2008b), these definitions can be classified in one of the following categories: (1) conventional economic perspective; (2) non-environmental degradation perspective; (3) integrational perspective, i.e. encompassing the economic, environmental, and social aspects; (4) inter-generational perspective; and (5) holistic perspective. Sustainability seen from the economic perspective is considered to confuse sustainability with economic viability, i.e. sustained growth and self-sufficiency (Lozano, 2008a), which howbeit should not be marginalized either.

The holistic perspective combines the integrational and inter-generational perspectives (Lozano, 2007) with the search for two dynamic and simultaneous equilibria: the first between the three pillars of sustainability; the second of continuum in a temporal manner. However, time planning, as a consideration of the future effects of today's actions and inactions as paramount, has often been relegated to a secondary role (Seghezzo, 2009). By modeling sustainability through multiplex networks, we implicitly address sustainability in a holistic manner, in that we attempt to take into account its different aspects, without omitting to subject them to time dynamics. From a broader perspective, our results should be viewed as a proof that multiplex networks can be put to good use to apprehend the topics relative to the sustainability of SES. To a lesser degree, our framework also succeeded in measuring the magnitude of spillover effects, which have previously been tested in Cherry et al. (2003). In order to validate or invalidate our theoretic statements, additional experimental works could be undertaken. In all cases, the model outcomes open an interesting debate on sustainability issues.

First, notwithstanding the risks of reform failure and abrogation, we do confirm the theoretical possibility to lead socio-ecological systems toward reforms that are considered as indispensable. We thus manage to exceed the limits imposed by the topology of the Fano plane. Second, achieving a worthwhile objective by reforming a multilayered architecture ought to be seen as transient, because the population of agents following the reform path is found to be non-stationary. Hence, monitoring and evaluating the reform process seem as important as setting it off on a path. Third, we do confirm the narrowness of the sustainability space, such as one depicted in the well-known Venn diagram. In other words, in presence of high likelihood of advancing an amendment, the sole proportional influence of layers constituting SES yields the maximum magnitude of efficiency of the knock-on effect. However, considering all aspects of sustainable development as of equal importance does not seem to be of clear evidence yet.

Indeed, good reforms offer critical insights on conflict between the various spheres of economy, society and ecology (Brennan, 2008). For example, Estapé-Dubreuil et al.

(2016) show that the criteria used in investment decision-making only depend on two out of three dimensions of sustainability. It implies that advancing two objectives requires sacrificing the third one. Timely, we can mention the topic of full employment, which is considered to be an obligatory macroeconomic objective to achieve sustainable development. Yet, full employment and ecological sustainability objectives seem to be in large conflict (Lawn, 2006). This is probably why, on the occasion of the 21st Conference of the Parties to the United Nations Framework Convention on Climate Change (COP21), a call for a deep change in mentalities has been made. Besides, by discriminating the roles to play by the three pillars underlying the SES setting, one may achieve sustainability, but at a cost of greater efforts, because the knock-on effect shall be moderately efficient. Provided that, in addition to the cost of monitoring the overall process of reform implementation, the sacrificed objective would need to be rehabilitated in the long-run, this type of strategy can be reasonably evaluated as economically unsound.

Despite its apparent abstractness, this work can be used easily to measure the impact of constraints under which the triple dividend effect (Tanner et al., 2015), while investing in disaster resilience, would take off. Carpenter et al. (2012) speak about general resilience as of the capacity of SES to transform in response to unfamiliar, unexpected and extreme shocks such as natural hazards. Even if a disaster does not occur, investing in resilience should provide evidence for three types of co-benefits, which are social protection by saving lives, economic growth by engaging in long-term investments and environmental benefits by avoiding environmental degradation. Nonetheless, building resilience at this scale requires to design and implement the right incentives. The last authors enumerate a list of conditions that enable the achievement of general resilience. Those include diversity, modularity, openness, reserves, feedbacks, nestedness, monitoring, leadership, and trust. Not only do our results support the indispensability of these qualitative criteria, but also provide a formalized cadre for conducting a quantitative analysis of resilience, from a perspective of interactions in multilayered networks, which is among the pressing challenges when it comes to incorporating reforms in complex systems, for the concept is hard to translate into measurable variables.[8]

To conclude, let us dwell on the price-regulating mechanisms and the environmental

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[8] This work could also be associated to what Sneddon et al. (2006) term deliberative democracy in a post-Brundtland world, in that it is built on decentralized decision-making and equal treatment of spheres composing sustainable development. In that case, the model enables to measure its efficiency.

pricing reforms. If we replace reforms by market-price fluctuations toward optimal prices for sustainable development (Pearce, 1988), in which prices observed on markets fully incorporate social costs and environmental externalities, a reform failure becomes the status quo on price levels as a result of improper price updates. The same goes with reform abrogation, which can then be interpreted as an impediment to market corrections inclusive of non-economic impacts. Should this be the case, the results of the model indicate that the pricing – without ever reaching stationarity in the long run – would benefit from equally considering market supply and demand along with the environmental repercussions of production and consumption, not forgetting the aspect of social cohesion with respect to the access to goods and services produced within the society. This statement seems to mirror that of Kahn (2015), who recalls the imperfect tradeoffs between economy, environment and equity. To a certain extent, it also ties up with the idea of making greater use of full-cost accounting (Richards, 1997) and that of shadow pricing (van Soest et al., 2006).

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