Exchange Networks with Stochastic Matching

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Abstract

We consider an exchange network with stochastic matching between the pairs of players and analyze the balanced dynamics of bargaining in such a market. The cases of homogeneous expectations, heterogeneous expectations and social preferences are studied. The results show that, in all three cases, the dynamics converge to the solution concept of balanced outcome or Nash bargaining solution, which is an equilibrium concept that combines notions of stability and fairness. In the two first scenarios, the numerical simulations reveal that the convergence to a fixed point stands at the value of the outside option. In the third scenario, the fixed point converges to the value of the outside option supplemented by the surplus share.

1 Introduction

Bargaining has received great attention from the economists (Nash, 1950; Binmore et al., 1986; Bulow and Klemperer, 1996). Nash bargaining, as a solution satisfying axioms of rationality in a non-cooperative game, has been previously studied with regard to vertical markets (Iozzi and Valletti, 2014), international relations (Jackson and Morelli, 2007), innovation partnerships (Pun and Ghamat, 2016), labor relations (Chakrabartia and Tangsangasaksri, 2011) or power division among stakeholders (Cachon and Lariviere, 2005). The Nash solution to a bilateral bargaining problem involves the determination of payoffs for each party with a specification of the disagreement point if the negotiation breaks down (Iozzi and Valletti, 2014). Whereas the case of bargaining between two players is now well understood, less is known about the possible outcomes of bargaining on a large scale such as networks.

Following the literature on exchange networks (Kleinberg and Tardos, 2008; Azar et al., 2009; Bayati et al., 2015), the latter being based on games on graphs, we consider players to occupy the nodes of a network, while the edges represent the trading partnerships, as a result of a matching mechanism, between the pairs of players. The previous works introduced the concept of bargaining dynamics and studied the convergence properties in the states of matched players defined over their allocations. Balanced dynamics assumes that players first agree on matching and then negotiate the values of the outcomes, which can be equal to or different from the well-known Nash equilibria. The balanced outcome for such a game is an equilibrium concept that combines the notions of stability, by consolidating the trading partnership, and of fairness, by equally splitting the trading surplus. Kleinberg and Tardos (2008) proved that if a network admits a stable outcome it also admits a balanced outcome, which can be seen as a form of generalization of the Nash bargaining solution (Bayati et al., 2015). If we now consider the process in which the partnership is concluded at random, this involves a process of stochastic matching, where the nodes are matched with some probability, implicitly involving the construction of a random graph.[1]

We consider an exchange network with stochastic matching between the pairs of players and analyze the balanced dynamics of bargaining in such a market. The cases of

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[1]The latter has been mostly studied, from the framework proposed by Gilbert (1959), such that an edge occurs independently with some non-null probability.

homogeneous expectations, heterogeneous expectations and social preferences are studied. The results show that, in all three cases, the dynamics converge rapidly to the solution concept of balanced outcome or Nash bargaining solution. In the two first scenarios, the numerical simulations reveal that the convergence to a fixed point stands at the value of the outside option. In the third scenario, the fixed point converges to the value of the outside option supplemented by the surplus share.

Section 2 provides a detailed description of the network model in the respective scenarios. Section 4 is devoted to illustrating simulation examples. Conclusive remarks are given in Section 5.

2   Model

Let us first introduce the deterministic framework that has been used in the literature so far. Consider a network of players spread across a graph $G = (V, E)$, where $V = \{v_1, ..., v_n\}$ denotes the set of nodes representing the players and $E = \{(u, v) \in V \times V\}$ is the set of edges or connections between those players.

Players u and v, for $u, v = 1, ..., n$ and $u \neq v$, connected through an edge (u, v) are allowed to play the bargaining game at time t, $\forall t \geq 0$. The game aims at splitting the value generated by their exchange. Before the negotiation between matched players takes place, let $w_{u,v}(t) : E \to \mathbb{R}_+^{V \times V}$ be the weight of an edge at time t measuring the aggregated value destined to negotiation. Each player can at most trade with one of its neighbors. When one exists, the game outcome is related to a matching $M(t) \subseteq E$ of G, which defines the pairs of players involved in exchanges at time t. When the game leads to a trade outcome, let a vector $\mathbf{x}(t) \in \mathbb{R}_+^V$ describe the allocations on V. For example, if $x_u(t)$ is the allocation of node u, the outcome $x_u(t) + x_v(t) = w_{u,v}(t)$, issued from an edge $(u, v) \in M(t)$, is the solution of the game at time t. Otherwise, $x_u(t) = 0$ for every unmatched node $u \notin M(t)$.

Let us then extend the original model by injecting stochasticity in the networked system. In order to compute the expected trade outcomes, we now assign choice probabilities to all the pairs of players. Consider $\mathbb{P}[(u, v) \in M(t)]$ to be the probability that two players decide to get involved in a negotiation, which implicitly puts a probability that an edge between u and v is chosen. Likewise, $\mathbb{P}[(u, v) \notin M(t)]$ represents the probability that

player v, as a potential counterpart of player u in the bargaining game, chooses a neighbor different from u. Denote by $n_G(u)$ the set of neighbors of player u, where $k \in n_G(u) \backslash v$. Therefore, we are faced with a random graph $G = (V, E, P)$, defined over the probability measure function $P : \mathcal{F} \to [0,1]$, with $\mathcal{F}$ the σ-algebra. The probability function assigns to each node the likelihood that a neighbor chooses it as a trading partner among the set of nodes. According to the methodology introduced by Beeri et al. (2004) and Safra et al. (2010), the probability functions are

$$\mathbb{P}[(u,v) \in M(t)] = \frac{[x_u(t) - x_v(t)]^\alpha}{\sum_{(u,k) \in M(t)} [x_u(t) - x_k(t)]^\alpha + w_{u,v}(t)^\alpha} \tag{1}$$

$$\mathbb{P}[(u,v) \notin M(t)] = \frac{w_{u,v}(t)^\alpha}{\sum_{(u,k) \in M(t)} [x_u(t) - x_k(t)]^\alpha + w_{u,v}(t)^\alpha} \tag{2}$$

$\forall t \geq 0$, for $u, v = 1, ..., n$ and $k \in n_G(u) \backslash v$ and $(u,v) \in E \backslash M(t)$. As for $\alpha \in [0,1]$, it represents the decay factor in the probabilistic method. Assuming a decay factor of 1 leads to certainty that u will choose v. Formally, the probability function is defined in terms of the difference in allocations between players u and v, and of the difference in allocations between player u and an alternative neighbor k, should the negotiation with the matching partner fail, reflecting the outside option. In parallel, the probability that players u and v do not match depends on the value at stake, before the realization of the exchange, as well as on the gap in allocations between player u and player k.

2.1 Homogeneous expectations

By means of identical decay factors $\alpha_u, \alpha_v \in [0,1]$, such that $\alpha_u = \alpha_v = \alpha$, let players u and v exhibit homogeneous expectations on the possibility to initiate a trading exchange. In this case, the expected outcome amounts to

$$\mathbb{E}[x_u(t) + x_v(t)] = \mathbb{E}[w_{u,v}(t)] \tag{3}$$
$$\Leftrightarrow [x_u(t) + x_v(t)]\mathbb{P}[(u,v) \in M(t)] = w_{u,v}(t)\mathbb{P}[(u,v) \notin M(t)]$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, from which we obtain the following weight in expectation

$$w_{u,v}(t) = [[x_u(t) + x_v(t)][x_u(t) - x_v(t)]^\alpha]^{\frac{1}{1+\alpha}} \tag{4}$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, $(u, v) \in E \backslash M(t)$. The notion of equilibrium that captures the rational play in the bargaining game resumes to a stable outcome. Such requirement, where the sum of the values is maximal, which implies a maximum weight matching (Shapley and Shubik, 1972), states that an unrealized exchange between two players cannot be better than the realized one. Put differently, the player cannot earn more by changing its trading partner. Formally,

$$w_{u,v}(t) \leq [[x_u(t) + x_v(t)][x_u(t) - x_v(t)]^\alpha]^{\frac{1}{1+\alpha}} \tag{5}$$

Each player also has an alternative option, which represents its expected value in case they disagree on how to split the value. Let $\mathbb{E}[\beta_u(t)]$ and $\mathbb{E}[\beta_v(t)]$ be the respective option values of players u and v such that[2]

$$
\begin{aligned}
\mathbb{E}[\beta_u(t)] &= \beta_u(t)\mathbb{P}[(u, v) \in M(t)] \\
&= \frac{[w_{u,v}(t) - x_v(t)][x_u(t) - x_v(t)]^\alpha}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^\alpha + w_{u,v}(t)^\alpha}
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
\mathbb{E}[\beta_v(t)] &= \beta_v(t)\mathbb{P}[(u, v) \in M(t)] \\
&= \frac{[w_{u,v}(t) - x_u(t)][x_u(t) - x_v(t)]^\alpha}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^\alpha + w_{u,v}(t)^\alpha}
\end{aligned}
\tag{7}
$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, $k \in n_G(u)\backslash v$ and $(u, v) \in E \backslash M(t)$.

The exchange yields an expected surplus of

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[2]Whereas, in the Nash game, the alternatives are given exogenously, the alternatives in the network bargaining game are given endogenously.

$$\mathbb{E}[s_{u,v}(t)] = \mathbb{E}[w_{u,v}(t)] - [\mathbb{E}[\beta_u(t)] + \mathbb{E}[\beta_v(t)]] \tag{8}$$

$$= \frac{w_{u,v}(t)^{1+\alpha} - [x_u(t) - x_v(t)]^{\alpha} [2w_{u,v}(t) - x_u(t) - x_v(t)]}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^{\alpha} + w_{u,v}(t)^{\alpha}}$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, $k \in n_G(u)\backslash v$ and $(u,v) \in E\backslash M(t)$.

While the stable outcome ensures that the outcome from a matched exchange is always preferred, a stable balanced outcome is an outcome in which the edge endpoints satisfy the Nash bargaining solution at any point of time (Azar et al., 2009). The latter posits that players split the surplus evenly between them, that is, the surplus of u over its alternative equals that of v over its own alternative. In expectation, the equilibrium values amount to

$$x_u^{\star}(t) = \mathbb{E}[\beta_u(t)] + \mathbb{E}[s_{u,v}(t)]/2 \tag{9}$$

$$= \frac{1}{2}\frac{[x_u(t) - x_v(t)]^{1+\alpha} + w_{u,v}(t)^{1+\alpha}}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^{\alpha} + w_{u,v}(t)^{\alpha}}$$

$$x_v^{\star}(t) = \mathbb{E}[\beta_u(t)] + \mathbb{E}[s_{u,v}(t)]/2 \tag{10}$$

$$= \frac{1}{2}\frac{w_{u,v}(t)^{1+\alpha} - [x_u(t) - x_v(t)]^{1+\alpha}}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^{\alpha} + w_{u,v}(t)^{\alpha}}$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, $k \in n_G(u)\backslash v$ and $(u,v) \in E\backslash M(t)$.

We know that a stable outcome corresponds to the relative inequality $x_u(t) + x_v(t) \geq w_{u,v}(t)$. Provided the values of $x_u^{\star}(t)$, $x_v^{\star}(t)$ and $w_{u,v}(t)$, we obtain

$$x_u^{\star}(t) + x_v^{\star}(t) \geq w_{u,v}(t) \tag{11}$$

$$\leq \frac{w_{u,v}(t)^{1+\alpha}[x_u(t) - x_v(t)]^{-\alpha}}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^{\alpha} + w_{u,v}(t)^{\alpha}}$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, $k \in n_G(u)\backslash v$ and $(u,v) \in E\backslash M(t)$. When the surplus is meant to be split equally, the disparity between the allocations goes to zero or $[x_u(t) - x_v(t)]^{\alpha} \to 0$. This implies that $x_u(t) + x_v(t) \leq \infty$, which is always verified. Thereby, the

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Nash solutions with homogeneous expectations prove to be stable. This leads us to the following proposition.

Proposition 1 *In exchange networks with stochastic matching and homogeneous expectations of players, the bargaining outcome as a game equilibrium is stable.*

Let us now prove that the dynamics converge to a fixed point. Following Muthoo (1999), we need to differentiate either $x_u^\star(t)$ or $x_v^\star(t)$ with respect to time. By ignoring the player $k \in n_G(u)\backslash v$, after setting an equal split, such that $x_u(t) = x_v(t)$, one obtains a differential equation in form of

$$\frac{\left[C(1+\alpha)w_{u,v}(t)^\alpha - \alpha w_{u,v}(t)^{\alpha-1}\right] w'_{u,v}(t)}{C^2} = 0 \tag{12}$$

where $C = \sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^\alpha + w_{u,v}(t)^\alpha$, the solution of which is an arbitrary constant related to the level of the unrealized exchange $w_{u,v}(t)$. In consequence, the Nash equilibrium solution converges as one runs the balancing dynamics. The second proposition ensues.

Proposition 2 *In exchange networks with stochastic matching and homogeneous expectations of players, the bargaining outcome as a game equilibrium converges to a fixed point.*

2.2 Heterogeneous expectations

Now consider divergent expectations of players u and v through the use of differentiated decay factors $\alpha_u, \alpha_v \in [0,1]$ such that $\alpha_u \neq \alpha_v$. The expected outcome is represented in form of a Diophatine equation (Yang, 2013), which yields

$$\mathbb{E}_u[x_u(t)] + \mathbb{E}_v[x_u(t)]x_v(t) = \mathbb{E}_{u\times v}[w_{u,v}(t)] \tag{13}$$
$$\Leftrightarrow x_u(t)\mathbb{P}_u[(u,v) \in M(t)] + x_v(t)\mathbb{P}_v[(u,v) \in M(t)] = w_{u,v}(t)\mathbb{P}_u[(u,v) \notin M(t)]\mathbb{P}_v[(u,v) \notin M(t)]$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, $k \in n_G(u)\backslash v$ and $(u,v) \in E\backslash M(t)$.

The use of the equation enables us to assign an expected weight of

$$w_{u,v}(t) \;=\; \frac{x_u(t)\mathbb{P}[(u,v)\in M(t)] + x_v(t)\mathbb{P}[(u,v)\in M(t)]}{\mathbb{P}_u[(u,v)\notin M(t)]\mathbb{P}_v[(u,v)\notin M(t)]} \tag{14}$$

$$= \; \left[x_u[x_u(t)-x_v(t)]^{\alpha_u}A + x_v[x_u(t)-x_v(t)]^{\alpha_v}B\right]^{\frac{1}{1+\alpha_u+\alpha_v}}$$

$\forall t \geq 0$, for $u,v = 1,...,n$, $k \in n_G(u)\backslash v$, $(u,v) \in E\backslash M(t)$ and $\alpha_u \neq \alpha_v$, where $A = \sum_{(u,k)\in M(t)}[x_u(t)-x_k(t)]^{\alpha_v}+w_{u,v}(t)^{\alpha_v}$ and $B = \sum_{(u,k)\in M(t)}[x_u(t)-x_k(t)]^{\alpha_u}+w_{u,v}(t)^{\alpha_u}$.

Accordingly, the stable outcome corresponds to

$$w_{u,v}(t) \leq \left[x_u[x_u(t)-x_v(t)]^{\alpha_u}A + x_v[x_u(t)-x_v(t)]^{\alpha_v}B\right]^{\frac{1}{1+\alpha_u+\alpha_v}} \tag{15}$$

As for the option values, they now amount to

$$\mathbb{E}_u[\beta_u(t)] \;=\; \beta_u(t)\mathbb{P}_u[(u,v)\in M(t)] \tag{16}$$

$$= \; \frac{[w_{u,v}(t)-x_v(t)][x_u(t)-x_v(t)]^{\alpha_u}}{\sum_{(u,k)\in M(t)}[x_u(t)-x_k(t)]^{\alpha_u}+w_{u,v}(t)^{\alpha_u}}$$

$$\mathbb{E}_v[\beta_v(t)] \;=\; \beta_v(t)\mathbb{P}_v[(u,v)\in M(t)] \tag{17}$$

$$= \; \frac{[w_{u,v}(t)-x_u(t)][x_u(t)-x_v(t)]^{\alpha_v}}{\sum_{(u,k)\in M(t)}[x_u(t)-x_k(t)]^{\alpha_v}+w_{u,v}(t)^{\alpha_v}}$$

$\forall t \geq 0$, for $u,v = 1,...,n$, $k \in n_G(u)\backslash v$, $(u,v) \in E\backslash M(t)$ and $\alpha_u \neq \alpha_v$.

The latter serve to compute the updated expression of the expected surplus

$$\mathbb{E}_{u\times v}[s_{u,v}(t)] \;=\; \mathbb{E}_{u\times v}[w_{u,v}(t)] - [\mathbb{E}_u[\beta_u(t)] + \mathbb{E}_b[\beta_v(t)]] \tag{18}$$

$$= \; \frac{w_{u,v}(t)^{1+\alpha_u+\alpha_v} - [w_{u,v}(t)-x_v(t)]\,[x_u(t)-x_v(t)]^{\alpha_u}A}{AB}$$

$$- \; \frac{[w_{u,v}(t)-x_u(t)]\,[x_u(t)-x_v(t)]^{\alpha_v}B}{AB}$$

$\forall t \geq 0$, for $u,v = 1,...,n$, $k \in n_G(u)\backslash v$, $(u,v) \in E\backslash M(t)$ and $\alpha_u \neq \alpha_v$, where $A = \sum_{(u,k)\in M(t)}[x_u(t)-x_k(t)]^{\alpha_v}+w_{u,v}(t)^{\alpha_v}$ and $B = \sum_{(u,k)\in M(t)}[x_u(t)-x_k(t)]^{\alpha_u}+w_{u,v}(t)^{\alpha_u}$.

Finally, the Nash bargaining solutions are equal to

$$
\begin{aligned}
x_u^\star(t) &= \mathbb{E}_u[\beta_u(t)] + \mathbb{E}_{u \times v}[s_{u,v}(t)]/2 \\
&= \frac{w_{u,v}(t)^{1+\alpha_u+\alpha_v} + [w_{u,v}(t) - x_v(t)][x_u(t) - x_v(t)]^{\alpha_u} A}{2AB} \\
&\quad - \frac{[w_{u,v}(t) - x_u(t)][x_u(t) - x_v(t)]^{\alpha_v} B}{2AB}
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
x_v^\star(t) &= \mathbb{E}_v[\beta_u(t)] + \mathbb{E}_{u \times v}[s_{u,v}(t)]/2 \\
&= \frac{w_{u,v}(t)^{1+\alpha_u+\alpha_v} - [w_{u,v}(t) - x_v(t)][x_u(t) - x_v(t)]^{\alpha_u} A}{2AB} \\
&\quad + \frac{[w_{u,v}(t) - x_u(t)][x_u(t) - x_v(t)]^{\alpha_v} B}{2AB}
\end{aligned}
\tag{20}
$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, $k \in n_G(u) \backslash v$, $(u, v) \in E \backslash M(t)$ and $\alpha_u \neq \alpha_v$, where $A = \sum_{(u,k) \in M(t)} [x_u(t) - x_k(t)]^{\alpha_v} + w_{u,v}(t)^{\alpha_v}$ and $B = \sum_{(u,k) \in M(t)} [x_u(t) - x_k(t)]^{\alpha_u} + w_{u,v}(t)^{\alpha_u}$.

Given that a stable outcome corresponds to $x_u(t) + x_v(t) \geq w_{u,v}(t)$, we have

$$
\begin{aligned}
x_u^\star(t) + x_v^\star(t) &\geq w_{u,v}(t) \\
\frac{w_{u,v}(t)^{(1+\alpha_u+\alpha_v)^2}}{(AB)^{1+\alpha_u+\alpha_v}} &\geq x_u[x_u(t) - x_v(t)]^{\alpha_u} A + x_v[x_u(t) - x_v(t)]^{\alpha_v} B
\end{aligned}
\tag{21}
$$

Despite divergent expectations, when the surplus is intended to be split equally, we have $[x_u(t) - x_v(t)]^{\alpha_u, \alpha_v} \to 0$. The left-sided expression thus has to be greater than or equal to zero, which is always verified. As a result, the Nash solutions with heterogeneous expectations prove to be stable.

Proposition 3 *In exchange networks with stochastic matching and heterogeneous expectations of players, the bargaining outcome as a game equilibrium is stable.*

By ignoring the player $k \in n_G(u) \backslash v$, the time-derivative, with $x_u(t) = x_v(t)$, yields the following differential equation

$$
\frac{(1 + \alpha_u + \alpha_v) w_{u,v}(t)^{\alpha_u+\alpha_v} w_{u,v}'(t)}{AB} = 0
\tag{22}
$$

where $A = \sum_{(u,k)\in M(t)}[x_u(t)-x_k(t)]^{\alpha_v}+w_{u,v}(t)^{\alpha_v}$ and $B = \sum_{(u,k)\in M(t)}[x_u(t)-x_k(t)]^{\alpha_u}+w_{u,v}(t)^{\alpha_u}$, the solution of which is an arbitrary constant related to the level of the unrealized exchange $w_{u,v}(t)$. Once again, the Nash equilibrium solution converges.

Proposition 4 *In exchange networks with stochastic matching and heterogeneous expectations of players, the bargaining outcome as a game equilibrium converges to a fixed point.*

2.3 Social preferences

It is reasonable to expect that some nodes are likely to have more bargaining power than others (Kanoria et al., 2010). Nevertheless, one might also think about the presence of social preferences or ex-ante preferences for equity (Charness and Rabin, 2002), such that a player decides to allocate a portion of its value to the other player; the hypothesis is that a player sacrifices a positive amount of its share to accommodate other player's well-being. Let $\phi \in [0,1]$ be the outcome share that player u keeps on the edge (u,v), so $1-\phi$ is the share that is offered to player v. In that case, we have

$$\mathbb{E}[[\phi x_u(t) + (1-\phi)x_v(t)] + x_v(t)] \quad = \quad \mathbb{E}[w_{u,v}(t)] \tag{23}$$
$$\Leftrightarrow [[\phi x_u(t) + (1-\phi)x_v(t)] + x_v(t)]\mathbb{P}[(u,v)\in M(t)] \quad = \quad w_{u,v}(t)\mathbb{P}[(u,v)\notin M(t)]$$

$\forall t \geq 0$, for $u,v = 1,...,n$ and $k \in n_G(u)\backslash v$. When $\phi = 1$, player u is only concerned by its own outcome. Otherwise, when $\phi = [0,1)$, it is also concerned by the allocation that player v gets. The stable outcome is in form of

$$w_{u,v}(t) \leq [[\phi x_u(t) + (1-\phi)x_v(t)] + x_v(t)][x_u(t) - x_v(t)]^\alpha]^{\frac{1}{1+\alpha}} \tag{24}$$

The corresponding option values are equal to

$$\mathbb{E}[\beta_u(t)] \quad = \quad \beta_u(t)\mathbb{P}[(u,v)\in M(t)] \tag{25}$$
$$= \quad \frac{[w_{u,v}(t) - x_v(t)][x_u(t) - x_v(t)]^\alpha}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^\alpha + w_{u,v}(t)^\alpha}$$

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$$\begin{aligned}
\mathbb{E}[\beta_v(t)] &= \beta_v(t)\mathbb{P}[(u,v) \in M(t)] & (26)\\
&= \frac{[w_{u,v}(t) - [\phi x_u(t) + (1-\phi)x_v(t)]][x_u(t) - x_v(t)]^\alpha}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^\alpha + w_{u,v}(t)^\alpha}
\end{aligned}$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, $k \in n_G(u)\backslash v$ and $(u,v) \in E\backslash M(t)$.

The expected surplus from exchange is now

$$\begin{aligned}
\mathbb{E}[s_{u,v}(t)] &= \mathbb{E}[w_{u,v}(t)] - [\mathbb{E}[\beta_u(t)] + \mathbb{E}[\beta_v(t)]] & (27)\\
&= \frac{w_{u,v}(t)^{1+\alpha} - [x_u(t) - x_v(t)]^\alpha[2w_{u,v}(t) - \phi x_u(t) - (2-\phi)x_v(t)]}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^\alpha + w_{u,v}(t)^\alpha}
\end{aligned}$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, $k \in n_G(u)\backslash v$ and $(u,v) \in E\backslash M(t)$.

At last, the equilibrium values with social preferences amount to

$$\begin{aligned}
x_u^\star(t) &= \mathbb{E}[\beta_u(t)] + \mathbb{E}[s_{u,v}(t)]/2 & (29)\\
&= \frac{1}{2}\frac{\phi[x_u(t) - x_v(t)]^{1+\alpha} + w_{u,v}(t)^{1+\alpha}}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^\alpha + w_{u,v}(t)^\alpha}
\end{aligned}$$

$$\begin{aligned}
x_v^\star(t) &= \mathbb{E}[\beta_u(t)] + \mathbb{E}[s_{u,v}(t)]/2 & (30)\\
&= \frac{1}{2}\frac{w_{u,v}(t)^{1+\alpha} - [(2\phi-1)x_u(t) - \phi x_v(t)][x_u(t) - x_v(t)]^\alpha}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^\alpha + w_{u,v}(t)^\alpha}
\end{aligned}$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, $k \in n_G(u)\backslash v$ and $(u,v) \in E\backslash M(t)$.

The stable outcome property yields

$$\begin{aligned}
x_u^\star(t) + x_v^\star(t) &\geq w_{u,v}(t) & (31)\\
\frac{w_{u,v}(t)^{(1+\alpha)^2}}{\sum_{(u,k)\in M(t)}[x_u(t) - x_k(t)]^\alpha + w_{u,v}(t)^\alpha} &\geq 0
\end{aligned}$$

$\forall t \geq 0$, for $u, v = 1, ..., n$, $k \in n_G(u)\backslash v$ and $(u,v) \in E\backslash M(t)$, with $[x_u(t) - x_v(t)]^{\alpha_u, \alpha_v} \to 0$, the expression has to be non-negative, which is always verified. The Nash solutions

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with social preferences are thus stable.

Proposition 5 *In exchange networks with stochastic matching and social preferences of players, the bargaining outcome as a game equilibrium is stable.*

By fixing $x_u(t) = x_v(t)$ and by ignoring the player $k \in n_G(u) \backslash v$, the time-derivative takes the form of the following differential equation

$$\frac{\phi \alpha w_{u,v}(t)^{2\alpha} w'_{u,v}(t)}{C^2} = 0 \tag{32}$$

where $C = \sum_{(u,k) \in M(t)} [x_u(t) - x_k(t)]^{\alpha} + w_{u,v}(t)^{\alpha}$, the solution of which is an arbitrary constant related to the level of the unrealized exchange $w_{u,v}(t)$ as well as proportional to the surplus share ϕ decided by player u. With social preferences, the Nash equilibrium solution converges as well.

Proposition 6 *In exchange networks with stochastic matching and social preferences of players, the bargaining outcome as a game equilibrium converges to a fixed point.*

3   Simulations

Based on the properties and conditions previously obtained, the aim of this section is to illustrate, through numerical simulations, the results previously obtained. Three examples are presented, each of which covers the balanced outcomes obtained with homogeneous expectations, heterogeneous expectations and social preferences. Consider the values of model parameters to be fixed at $w_{u,v}(0) = 10.00$, $x_u(0) = 5.10$, $x_v(0) = 4.90$, such that the initial allocation is slightly unbalanced at $t = 0$.

3.1   Homogeneous expectations

Fig. 1 depicts the levels of allocations with homogeneous expectations. Despite the initial disparity between the allocations, we observe that the Nash bargaining solutions, that is $x_u^{\star}$ and $x_v^{\star}$, lead to an equal split of surplus, or a stable outcome, for all levels of α. As expected, the identical values of $\{5, 5\}$ are obtained at $\alpha = 1$. In addition to the validity
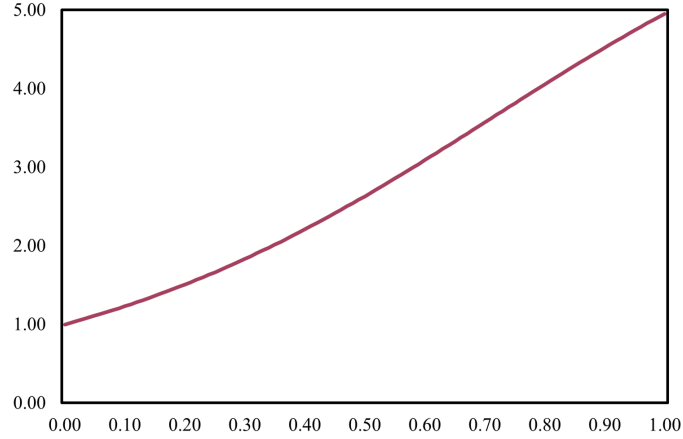
Figure 1: Nash bargaining solutions $(x_u^\star, x_v^\star)$ with homogeneous expectations. The x-axis corresponds to the decay parameter (α). The y-axis denotes the equilibrium values.

of Proposition 1, we also see that the continuum of equilibrium values takes the form of an s-shaped distribution function, with an inflection point at the mean coordinates $(0.5, 2.5)$.

For illustrative purposes, Fig. 2 represents the levels of option values. We denote that, given the initial unbalance, the convergence of alternative values takes place as $\alpha \to 1$. Even if the equilibrium values tend to 1 when the expectations head toward certainty, the options always remain strictly positive. Nevertheless, the continuum of values is subject to a monotonic convex decrease, which coincides with the rise in the Nash bargaining solutions previously described. And yet, the fall in option values is neither proportional nor symmetric to the increase in equilibrium values. This can be justified through the equal split of surplus in the Nash outcomes.
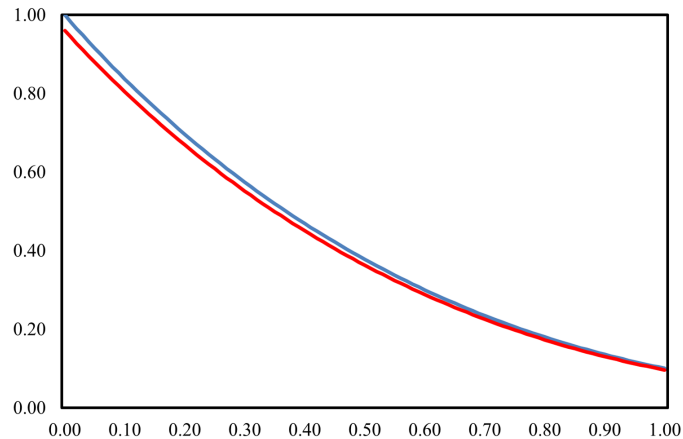

Figure 2: Option values (β_u, β_v) with homogeneous expectations. The x-axis corresponds to the decay parameter (α). The y-axis denotes the alternative values.

The convergence dynamics of Nash equilibria is outlined in Fig. 3, which has been simulated from the inflection point previously described. The results show a drop in expected values at early time steps, which then stabilize, from $t = 12$, at the levels observed with $\alpha = 0$. Thereby, not only do we confirm Proposition 2, but also find that the long-term Nash equilibria are positioned at a level where the expectation of being matched is zero. This implies that, in the long-run, the convergence of Nash bargaining solutions occurs at the levels of option values, which are the expected values in case the players fail to agree: a property that has been found through the convergence toward a constant equal to the value of the unrealized exchange.
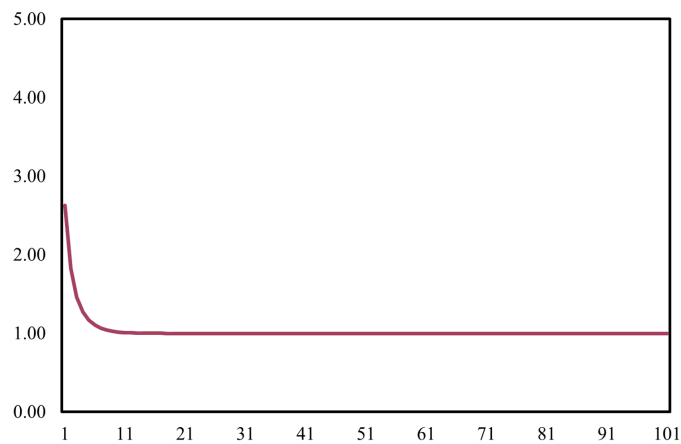

Figure 3: Convergence dynamics of Nash equilibra $(x_u^\star, x_v^\star)$ for $\alpha = 0.50$. The x-axis corresponds to the timeline (t). The y-axis denotes the evolution of equilibrium values as a function of time $(x_u^\star(t), x_v^\star(t))$.

Result 1 *In exchange networks with stochastic matching and homogeneous expectations of players, the bargaining outcome as a game equilibrium is stable and converges to the fixed value of the outside option.*

3.2   Heterogeneous expectations

Figs. 4 and 5 illustrate the levels of allocations with heterogeneous expectations. As can be noticed, the Nash bargaining solutions, that is $x_u^\star$ and $x_v^\star$, rise up to the levels of $\{5, 5\}$ when both decay parameters equal 1. The respective values of equilibria are found to be similar at low values of α_u and α_v; they are identical at levels close to certainty. We thus validate Proposition 3.
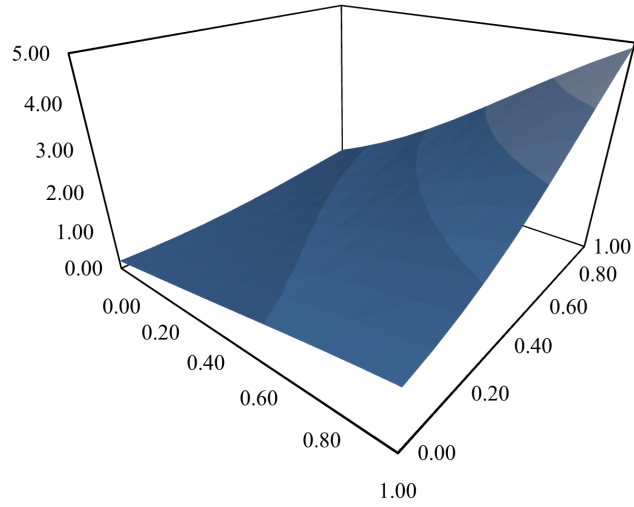
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Figure 4: Nash bargaining solution $(x_u^\star)$ with heterogeneous expectations. The left-sided x-axis corresponds to the decay parameter of player u (α_u). The right-sided y-axis is the decay parameter of player v (α_v). The z-axis denotes the equilibrium values.
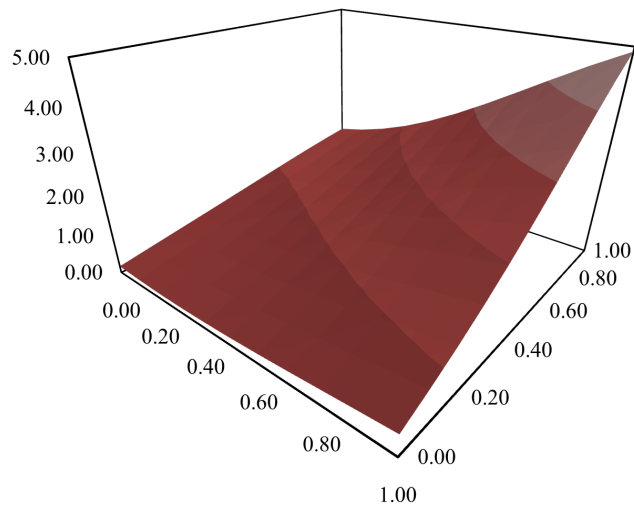

Figure 5: Nash bargaining solution $(x_v^\star)$ with heterogeneous expectations. The left-sided x-axis corresponds to the decay parameter of player u (α_u). The right-sided y-axis is the decay parameter of player v (α_v). The z-axis denotes the equilibrium values.

The convergence dynamics in case of heterogeneous expectations, where $\alpha_u = 0.60$ and $\alpha_v = 0.70$, is pictured in Fig. 6. Like in the previous case, we observe a decrease in expected values at the early time interval, which then freeze, from $t = 10$, at the level observed with $\alpha = 0$. We validate Proposition 4, and corroborate the convergence of Nash bargaining solutions toward a constant relative to the unrealized exchange.

Figure 6: Convergence dynamics of Nash equilibra $(x_u^\star, x_v^\star)$ for $\alpha_u = 0.60$ and $\alpha_v = 0.70$. The x-axis corresponds to the timeline (t). The y-axis denotes the evolution of equilibrium values as a function of time $(x_u^\star(t), x_v^\star(t))$.

Result 2 *In exchange networks with stochastic matching and heterogeneous expectations of players, the bargaining outcome as a game equilibrium is stable and converges to the fixed value of the outside option.*

3.3 Social preferences

With respect to the scenario dedicated to social preferences, Figs. 7 and 8, which have been differentiated – due to the inversely proportional share of allocations – by reversing the values of the abscissa, show equivalent patterns of equilibrium values of both players. The simulation results thus confirm Proposition 5. Another interesting outcome provided by the numerical simulations is that, in expectation, both equilibrium values tend to $\{10, 10\}$ when the share that player u keeps to itself heads toward 0. This implies that, with stochastic matching and social preferences, both players obtain levels equal to the expected surplus from the trading exchange.

This last result validates the soundness of the model based on reciprocity preferences developed by Rabin (1993) and extended by Levine (1998). According to those, two

players increase each other's payoffs when they expect to be treated by their partner fairly. In other words, given that player v benefits from the social concern of player u, it behaves toward its counterpart in a similar manner. Therefore, as $\alpha \to 0$, they both end up having expected allocations that attain the amount of the total surplus.
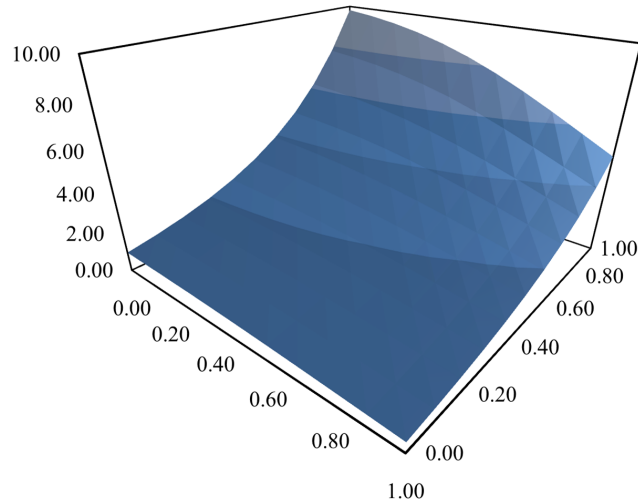

Figure 7: Nash bargaining solution $(x_u^\star)$ with social preferences. The left-sided x-axis corresponds to the share of player u (ϕ). The right-sided y-axis is the decay parameter of player u (α_u). The z-axis denotes the equilibrium values.

In case of social preferences, based on a probability of match of 0.5, with a level of share fixed to $\phi = 0.80$, the convergence dynamics observed in Fig. 9 validates the statement of Proposition 6. After an early swinging in respective allocations, both players obtain, from $t = 4$, identical equilibrium values throughout the time span. By that, because of reciprocal preferences, the Nash bargaining solutions head toward a constant which – due to the median expectation – is less than proportional to the equal split of the total surplus, but greater than the outside option, for it amounts a level almost twice higher.

Result 3 *In exchange networks with stochastic matching and social preferences of players, the bargaining outcome as a game equilibrium is stable and converges to the fixed value of the outside option supplemented by the surplus share.*

4   Conclusion

This work proves that balanced outcomes, as a result of interacting on exchange networks or assignment markets, can be achieved in case of stochastic matching, which, to the best
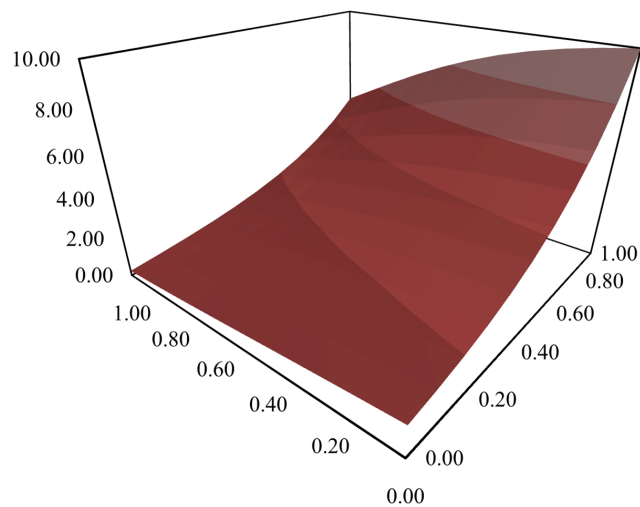
Figure 8: Nash bargaining solution $(x_v^\star)$ with social preferences. The left-sided x-axis corresponds to the share of player v $(1-\phi)$. The right-sided y-axis is the decay parameter of player v (α_v). The z-axis denotes the equilibrium values.

Figure 9: Convergence dynamics of Nash equilibria $(x_u^\star, x_v^\star)$ for $\alpha = 0.50$ and $\phi = 0.80$. The x-axis corresponds to the timeline (t). The y-axis denotes the evolution of equilibrium values as a function of time $(x_u^\star(t), x_v^\star(t))$.

of our knowledge, has not been considered in the study of matching markets. The numerical simulations unveil that the bargaining between players leads rapidly to a convergent fixed point, which is shown to be identical, both in values and time quantiles, in the cases of homogeneous and heterogeneous expectations. The scenario in which social preferences are examined yields the same conclusion, with a velocity toward convergence even more pronounced. Another interesting outcome lies in the validity of reciprocal preferences, which happen to generate the highest expected outcomes in balanced dynamics between the three cases in point.

Unlike the assumption encountered in the literature, where agents are unaware of the best alternatives of their neighborhood, we can consider our players to be pseudo-strategic, for the probability of matching also depends on the difference between the allocations, which can only be measured were the match already in place. Indeed, when players compare their surplus share with the outcome obtainable in absence of exchange, they can deduce the expectation of their neighbor, by observing the complementary probability, which provides them with some information on its best alternative. This attribute, coming from the model construction itself, explains the stability of outcomes along the timeline and provides consistency between our results and those of Rubinstein (1982), who justifies the pairwise Nash bargaining solution through the comparison of alternatives.

In consequence, our results do have a game-theoretic meaning, both in terms of alternatives and reciprocity toward equity. Nevertheless, we do not provide the informational assumption of common knowledge, which categorizes it among the models of incomplete information and bounded rationality. Even if these characteristics prevent a stable allocation to be established immediately (Bolle and Otto, 2016), allocations are known to be adoptable in a stochastic process with repeated re-matching, in which the core allocations with non-negative surpluses are found to be stochastically stable (Nax and Pradelski, 2015).

Despite the fact that our work can been considered as an extension of or a complement to the existing literature, for it regards the balancing dynamics in a random-graph framework, complementary works on stochastic chocs and risk-transfer allocations ought to be conducted.

Acknowledgment

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