Second price all-pay auctions, how much money do players lose?

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Preliminary version

Abstract

The paper studies second price all-pay auctions -static wars of attrition- in a new way, based on class room experiments and Kosfeld, Droste and Voorneveld's (2002) best reply matching equilibrium. Two players fight over a prize of value V, have a budget M, submit bids lower or equal to M. Both pay the lowest bid and the prize goes to the highest bidder; in case of a tie, each player gets the prize with probability 1/2. The mixed Nash equilibrium distribution has a special shape, with an atom on M and decreasing probabilities from bid 0 to a threshold bid.

Yet the students' probability distributions in the class room experiments are strikingly different (atom on bid 0, very low probabilities on small bids different from 0, and significant probabilities on bid M and other high bids). In fact, the students' behaviour fits with best reply matching, an ordinal logic according to which, if bid A is the best response to bid B, and if B is played with probability p, then A is also played with probability p.

The problem is that with a best reply matching behaviour- which better fits with real behaviour than mixed Nash equilibria-, second price all-pay auctions may become dangerous. In the mixed Nash equilibrium, the expected payoff is never negative, in the best reply matching equilibrium, both players may lose money. In the paper, we study best reply matching equilibria for different values of M and V and we focus on the case with M large (in comparison to V). In that case, players lose on average $1/12^{th}$ of their budget. We also talk about possible bifurcations in the bidding behaviour. We argue that our results give insights into how to regulate games to avoid pathological gambling with a huge waste of money.

Keywords : second price all-pay auction – war of attrition – best reply matching – classroom experiment

JEL classification: C72, D44

1. Introduction

The paper studies second price all-pay auctions in a new way, based on class room experiments and Kosfeld, Droste and Voorneveld's best reply matching equilibrium (2002). Whereas a lot has been said on first price all-pay auctions, there are only few papers with experiments on second price all-pay auctions, which are equivalent to static wars of attrition (see Hörisch and Kirchkamp (2010) and Dechevaux, Kovenock and Sheremeta (2015) for experiments with this class of games). The second price all-pay auction studied in this paper goes as follows: two players fight over a prize of value V, have a budget M, and

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simultaneously submit bids lower or equal to M. Both pay the lowest bid and the prize goes to the highest bidder; in case of a tie, each player gets the prize with probability 1/2.

The mixed Nash equilibrium distribution of this game is known to have a special shape, with an atom on M and decreasing probabilities from bid 0 to a given threshold bid. Yet this shape is strikingly different from the ones obtained in our class room experiments (an experiment with 116 students, another with 109 students). The students bid 0 with a high probability, assign very low probabilities to small bids different from 0, and significant probabilities to M and other high bids. The students' and the Nash equilibrium probability distributions are so different that we can't conclude on overbidding or underbidding in comparison with the Nash equilibrium behaviour. This observation is partly shared by Hörisch and Kirchkamp (2010); whereas overbidding is regularly observed in first price all-pay auction experiments (see for example Gneezy and Smorodinsky (2006) and Lugovskyy et al. (2010)), Hörisch and Kirchkamp (2010) establish that overbidding is less obvious in second price all-pay auction experiments and that underbidding seems to prevail in sequential war of attrition experiments. In our class room experiments, the students' behaviour fits with best reply matching, a behaviour studied by Kosfeld et al. (2002), according to which, if bid A is the best response to bid B, and if B is played with probability p, then A is also played with probability p. Mixed Nash equilibria and best reply matching equilibria follow a different logic: whereas Nash equilibria probabilities are calculated so as to equalize the payoffs of the bids played at equilibrium, best reply matching probabilities just aim to match best responses, each bid being played as often as the bid to which it is a best reply.

Yet the best reply matching logic, which better fits with real behaviour, brings into light that second price all-pay auctions may be dangerous. In fact, second price all-pay auction games are like casino games: when you bid - you put money on the table- you have no idea of how much money you will get or lose. Unlike what happens in a first price all-pay auction, where you pay what you bid, in a second price all-pay auction, you pay the lowest bid, which may be your bid or the opponent's one. M, in particular, can play a bad role, in that nobody can bid more. M, even if it is large, much larger than V, can be seen as the bid to choose if one wants to win the prize, and possibly –but not surely- pay a low amount. Second price all-pay auctions, as long as you work with the mixed Nash equilibrium, are not dangerous, at least as regards the expected payoffs, which are null (in the continuous game) or positive (in the discrete game); this automatically follows from the way the mixed Nash equilibrium is calculated: each played bid leads to the same payoff, necessarily higher or equal to the payoff obtained with bid 0 (which is never negative). This is no longer true with a best reply matching logic, which may lead to play high bids too often, in that they are best responses to low bids.

In the paper, we study best reply matching equilibria for different values of M and V and we namely focus on the case with M large (in comparison to V). In that case, players lose on average $1/12^{th}$ of their budget. We also talk about focal points and possible bifurcations in the bidding behaviour. Our results may give insights into how to regulate games to avoid pathological gambling with a huge waste of money.

Section 2 gives the pure and mixed Nash equilibria for continuous second price all-pay auction games. It talks about the strategic structure of the game and the special shape of the

mixed Nash equilibrium probability distribution. It also links mixed Nash equilibria in the continuous game to mixed Nash equilibria in the discrete game, where M, V and the bids are integers. In section 3, we present two class room experiments and comment on the students' behaviour. In section 4, we present Kosfeld et al.'s best reply matching equilibrium, which builds on Bernheim (1984) and Pearce's(1984) notion of rationalizability, and we compare the mixed Nash equilibrium philosophy with the best reply matching philosophy. We namely focus on the meaning of probabilities in both concepts. We also present a generalized best reply matching philosophy, which provides a greater degree of flexibility in choosing best responses in case of several best responses to a given strategy. In section 5, we compare the student's behaviour to best reply matching equilibria and generalized best reply matching equilibria. The convergence sheds light on the structure of best responses in a second price all-pay auction with a limit amount M to bid, and how this structure leads to the equilibrium. But it also brings into light that some best responses are more focal than others, namely M, V and 0, and how these focal values impact the equilibrium payoffs. We namely talk about the role of M, and the possible switch to other focal values when M becomes large. In section 6, we study the best reply matching equilibria in the general case. We talk about the atom on bid 0, study the best reply matching equilibrium distribution for M=2V and M=V, and focus on the loss of payoffs ($1/12^{th}$ of M) when M is very large compared to V. Section 7 concludes on the role of M, on the incentives to limit the ratio M/V, on the possibility of bifurcations in the players' behaviour and on the possibility to delete M.

2. Nash equilibria in continuous and discrete second price all-pay auctions

Two players have a limit budget M. They fight over a prize of value V. Each player i submits a bid b_i, i=1,2 lower or equal to M. The prize goes to the highest bidder but both bidders pay the lowest bid. In case of a tie, the prize goes to each bidder with probability ½.

The second price all-pay auction game is often compared to a static war of attrition in continuous time, where each player has to choose a time t in the interval [0,1] to leave the game (1 plays the role of M); staying in the game is costly (the cost increases in time) but, as soon as one player leaves the game, the game stops and the other gets the prize (this amounts to saying that if player i bids less than player j, player j gets the prize and both pay b_i) (see for example Hendricks, Weiss and Wilson (1988) for the war of attrition with the interval of time).

This game is known to have a lot of intuitive asymmetric Nash equilibria, one player bidding nothing (0), the other bidding V or more. In some way, if the first player fears to lose money, whereas the second is a hothead, the first player bids 0, whereas the second can afford bidding M (even if M is much larger than V), given that he will pay nothing thanks to the cautious behaviour of the first player. Second price all-pay auctions are rightly seen as dangerous games with great opportunities: when a hothead meets another one, both hotheads lose a lot of money, but if he meets a cautious one, then he gets V without paying anything. And if a cautious player meets another cautious one, each player gets the expected amount V/2 without paying anything.

The game has common points with card games like poker games, who also confront opposite kinds of behaviour. So, in poker games, sharks and gamblers are aggressive in that they bet and raise, whereas rock and fishes are passive (they may fold immediately). Yet things are different, even without taking into account the value of the cards. In contrast to our second price all-pay auction game, in a poker game, an aggressive player (a hothead) can't get a lot of money when faced to a passive (cautious) one –because the latter folds immediately, so there is not much money on the table- and the same is true for two passive (cautious) players. And if two gamblers meet, only one of them will lose a lot of money. So, in some way, the second price all-pay auction is more dangerous than poker for hotheads (especially if M is much larger than V), and more interesting for cautious players (if, in case of tie, the prize goes to each player with probability ½).

Things become less intuitive when turning to the unique symmetric Nash equilibrium (NE) of this game, which is a mixed NE. The second price all-pay auction is equivalent to the static war of attrition, so the mixed NE has the same structure than the one already given in Hendricks, Weiss and Wilson (1988).

Folk result
All the bids in [M-V/2, M[are weakly dominated by M.
The symmetric mixed Nash equilibrium in the continuous game is given by: b is played with probability f(b)db, with f(b)= $e^{-b/V}/V$ for b in[0, M-V/2], M is an atom played with probability f(M)= 1-F(M-V/2) = $e^{1/2-M/V}$, and b in]M-V/2,M[is played with probability 0, where f(x) and F(x) are the density probability distribution and the cumulative probability distribution.

Proof see Appendix 1

Corollary of the folk result:
The symmetric mixed Nash equilibrium bidding function in the continuous game without a limit budget is given by: f(b)= $e^{-b/V}/V$ for b in[0, +∞[.

Proof see Appendix 2

Let us comment on these equilibria. Figures 1a and 1b give the general form of the probability distribution, for V=3 and M=6 (Figure 1a), for V=15 and M=30 (Figure 1b).

Let us talk about the atom on M and the probability $e^{1/2-M/V}$ assigned to M. The fact that M is an atom is rather intuitive, given that M is a focal point with a special property. Given that nobody can play more than M, given that many players bid less, a player is sure, when he bids M, to get the prize with a high probability and to most often pay less than V (especially if M-V/2 <V). In this case, bidding M leads to a negative payoff only if the opponent plays M too. The probability $e^{1/2-M/V}$ assigned to M leads to three remarks.

First, it is decreasing in M for a fixed value of V, which ensures a continuity between the game with a limit budget M going to infinity and the model without limit budget. In fact, whether the players have or not a limit budget M, they bid b in [0, M-V/2] with the same probability f(b) = $e^{-b/V}/V$. If there is a limit budget M, then f(M) = $\sum_{k=M-V/2}^{\infty} f(b_k)$, where f($b_k$)= $e^{-b_k/V}/V$, the probability assigned to b_k in the model without limit budget. So M

4

focuses the probabilities assigned to each bid higher than M-V/2 in the model without limit budget.

Figure 1a

Figure 1b
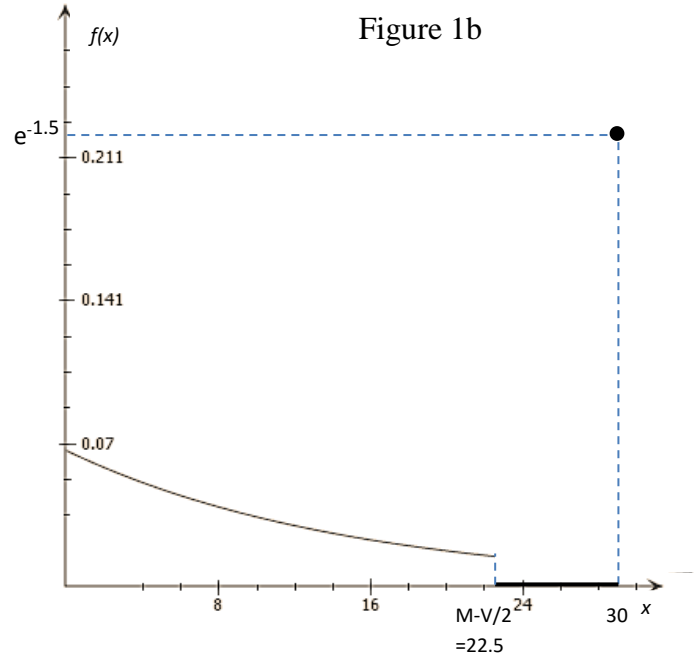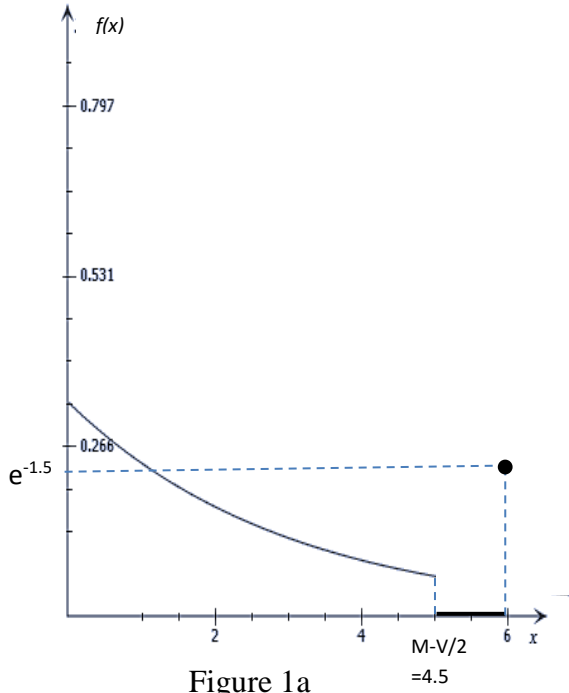
Second, $e^{1/2-M/V}$ may be large, especially when M is close to V, but it fast decreases when the ratio M/V grows (for M=V, f(M)=0.607, for M/V=1.5 f(M) = 0.368, for M/V=2 f(M)=0.223 but for M/V=5 f(M) = 0.011).

Third, the link between the probabilities assigned to M and 0 is far from being intuitive. There is no atom on bid 0, and f(0)db = db/V, so doesn't depend on M. Moreover, when V and M become large, but M/V remains constant, f(0) goes to 0 whereas f(M) remains constant. So for example, in Figures 1a and 1b, f(M)= $e^{-1.5}$ = 0.223 because 6/3=30/15= 2, but f(0)=0.333 for V=3 and M= 6 and f(0)= 0.067 for V=15 and M=30.

Let us talk about the probabilities assigned to the other bids, from 0 to M-V/2 (from 0 to ∞ when there is no limit M). As already observed, with or without limit budget, each bid b in [0, M-V/2] ([0, ∞[), is played with probability f(b)db = $(e^{-b/V}/V)$db. So the probability decreases in b, and the curves become flatter when V increases. It seems rather intuitive to bid 0 with a higher probability, in that bidding 0 never leads to lose money. But it is difficult to find an intuitive explanation for the decreasing probabilities.

Let us now draw attention to the fact that in our class room experiments, second price all-pay auctions are discrete games: M, V and b are integers, so the bid increment is equal to 1. As far as we know few has been said on the mixed NE in second price all-pay auction discrete games. So it matters to establish that the mixed NE in the discrete game is close the NE in the continuous game. We establish the convergence (under some conditions) in a companion paper (Umbhauer 2017).

We call q_i the probability a player assigns to bid i (so i goes from 0 to M-V/2-1 if V is even, to M-V/2-0.5 if V is odd).

> ### Result 2 (Umbhauer 2017)
> *When V is odd, q_i decreases in i, i being an integer in [0, M-V/2-0.5]. When V becomes large, the discrete Nash equilibrium probability distribution converges to the bidding density function in the continuous game.*
> *For V even, this convergence is only true for the sums q_i+q_{i+1} (which goes to f(i)+f(i+1)), i from 0 to M-V/2 -1 .*

Proof See Umbhauer 2017

Let us illustrate these results with some examples:

In Figures 2a and 2b, we give the NE probabilities for V=3 and M=5, and for V=30 and M=60 (bids going from tens to tens)[2], two games we study in the next section.
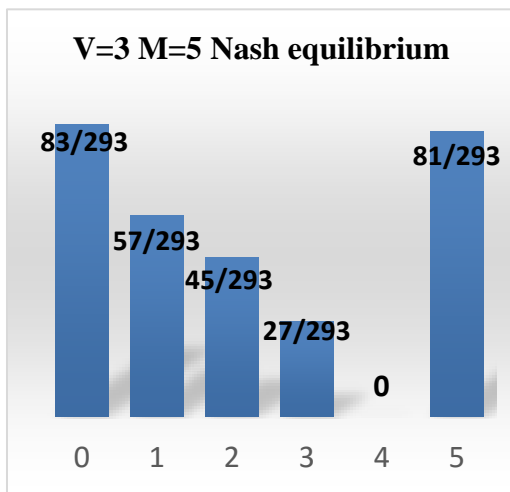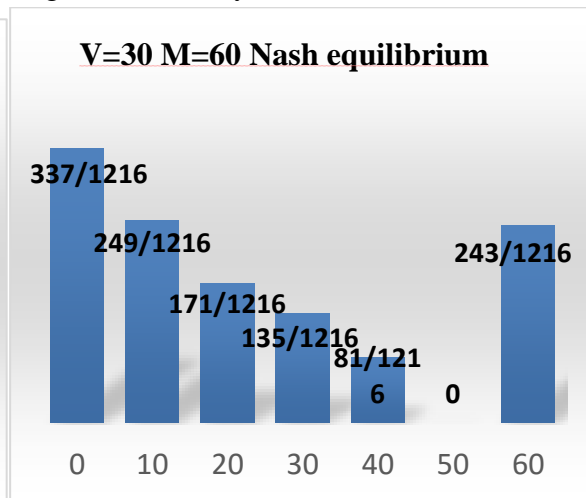

Figure 2a

Figure 2b

For the odd value V=9 and M=12 we get the NE probabilities in Figure 3a.
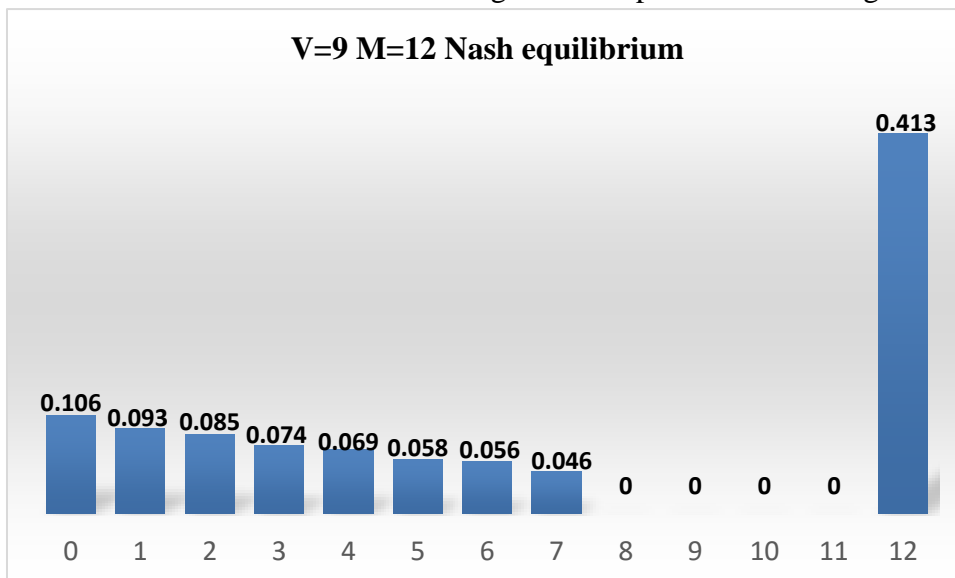

Figure 3a

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[2] This game is isomorphic to the game V=3, M=6, the bids being integers.

So the three cases exhibit a strong convergence as regards the shapes of the probability distributions in the discrete and in the continuous games. For V=9 and M=12, there is also a convergence in the values of the probabilities, in that the q_i get close to the f(i) for i from 0 to 12.

For the even value V=8 and M=12, we get the probabilities in Figure 3b .
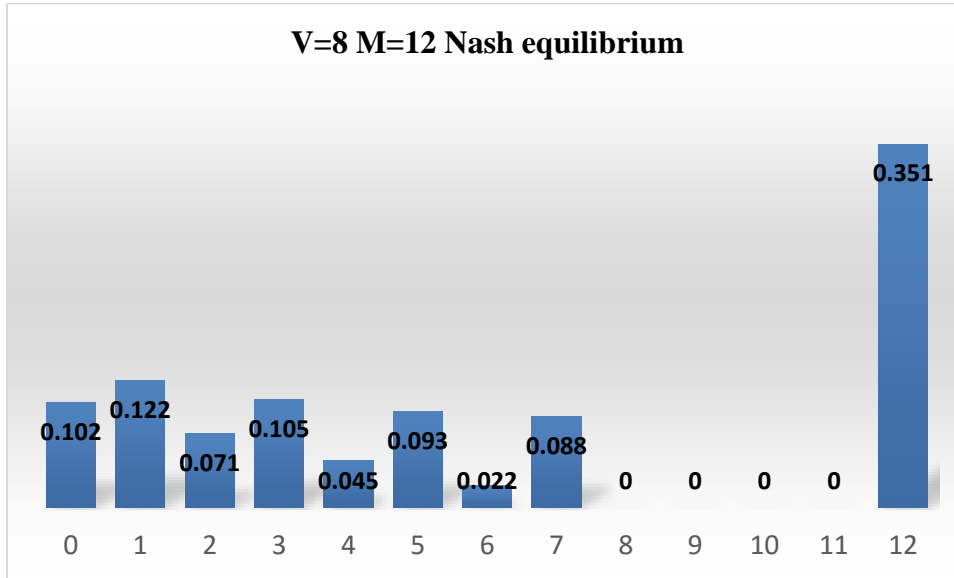

V=8 M=12 Nash equilibrium

Figure 3b

Here the discrete probabilities q_i do not conform with the smoothly decreasing exponential function f(i), and there is no convergence between the q_i and f(i), except for f(12) and q_{12}. Yet summing the probabilities two by two leads to a strong convergence in that q_i+q_{i+1} is very close to f(i)+f(i+1), for i from 0 to 10.

So, either directly, or by summing the probabilities two by two, we get a good convergence between the discrete model and the continuous one, which, in some degree, is not astonishing given that the logic of the calculations is the same in the discrete and in the continuous second price all-pay auction games. Yet, to be in a context more conform to the continuous NE equilibrium, it is better to choose V odd in experiments.

3. Class room experiments

In the first classroom experiment[3], 116 L3 students, i.e. undergraduate students in their third year of training, played the second price all-pay auction game in matrix 1, with V=3, M=5 and the possible bids 0, 1, 2, 3, 4 and 5. In the second classroom experiment, 109 L3 students played the second price all-pay auction game in matrix 2, with V=30, M=60 and the possible bids 0, 10, 20, 30, 40, 50 and 60. This second game is isomorphic to the game with V=3, M=6, the bids being 0, 1, 2, 3, 4, 5 and 6 (the payoffs have just to be divided by 10) –so we work with an odd V, despite V=30.

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[3] This experiment has also been partly studied in Umbhauer (2016).

The games were played during game theory lectures and all the students knew what is a normal form game. So they had no difficulty to understand the two games, and the meaning of the normal forms in matrix 1 and matrix 2. And the students had the matrices in front of them while choosing their bid. Let us add that the first game was played by students trained in Nash equilibria and dominance. By contrast, the second game was played by novice students with no training in Nash equilibria and dominance.

		Player 2					
		0	1	2	3	4	5
	0	(6.5,6.5)	(5,8)	(5,8)	(5,8)	(5,8)	(5,8)
	1	(8,5)	(5.5,5.5)	(4,7)	(4,7)	(4,7)	(4,7)
Player 1	2	(8,5)	(7,4)	(4.5,4.5)	(3,6)	(3,6)	(3,6)
	3	(8,5)	(7,4)	(6,3)	(3.5,3.5)	(2,5)	(2,5)
	4	(8,5)	(7,4)	(6,3)	(5,2)	(2.5,2.5)	(1,4)
	5	(8,5)	(7,4)	(6,3)	(5,2)	(4,1)	(1.5,1.5)

Matrix 1 V=3, M=5

		Player 2						
		0	10	20	30	40	50	60
	0	(75, 75)	(60, 90)	(60, 90)	(60, 90)	(60, 90)	(60, 90)	(60, 90)
	10	(90, 60)	(65, 65)	(50, 80)	(50, 80)	(50, 80)	(50, 80)	(50, 80)
	20	(90, 60)	(80, 50)	(55, 55)	(40, 70)	(40, 70)	(40, 70)	(40, 70)
Player1	30	(90, 60)	(80, 50)	(70, 40)	(45,45)	(30, 60)	(30, 60)	(30, 60)
	40	(90, 60)	(80, 50)	(70, 40)	(60, 30)	(35, 35)	(20,50)	(20,50)
	50	(90, 60)	(80, 50)	(70, 40)	(60, 30)	(50, 20)	(25,25)	(10,40)
	60	(90, 60)	(80, 50)	(70, 40)	(60, 30)	(50, 20)	(40,10)	(15,15)

Matrix 2 V=30, M=60

The students' way to play is given in table 1.

V=3, M=5 bids	Nash equilibrium probabilities	Students frequencies of the bids		V=30,M=60 bids	Nash equilibrium probabilities	Students frequencies of the bids
0	28.3%	38%		0	27.7%	33%
1	19.5%	9%		10	20.5%	5.5%
2	15.4%	1.5%		20	14.05%	2.8%
3	9.2%	20.5%		30	11.1%	21.1%
4	0	16%		40	6.65%	4.6%
5	27.6%	15%		50	0%	5.5%
				60	20%	27.5%

Table 1

The students' way to play is reproduced in Figures 4a and 4b, juxtaposed to the Nash equilibria in Figures 2a and 2b, the probabilities being replaced by an equivalent number of students.
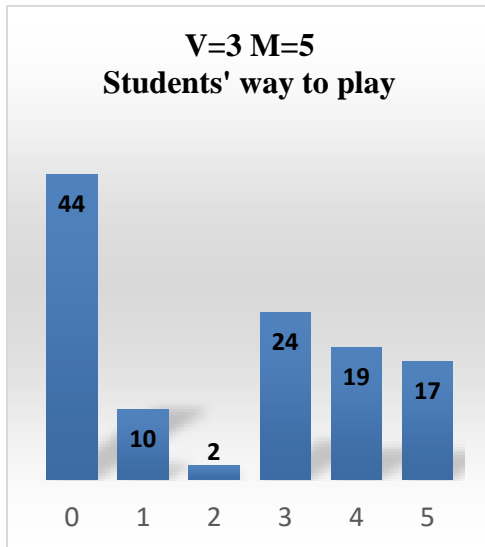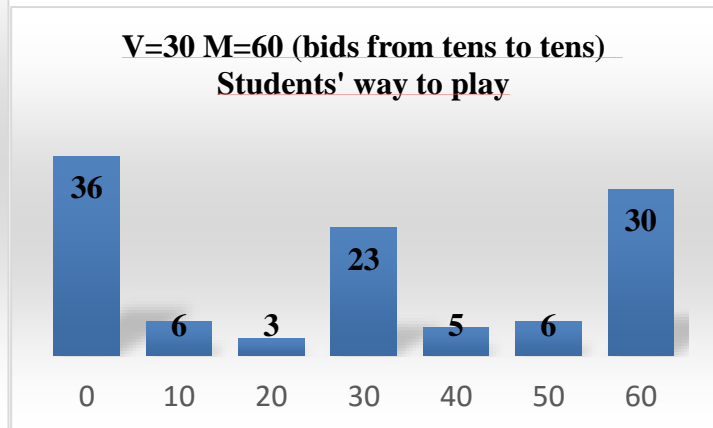
V=3 M=5
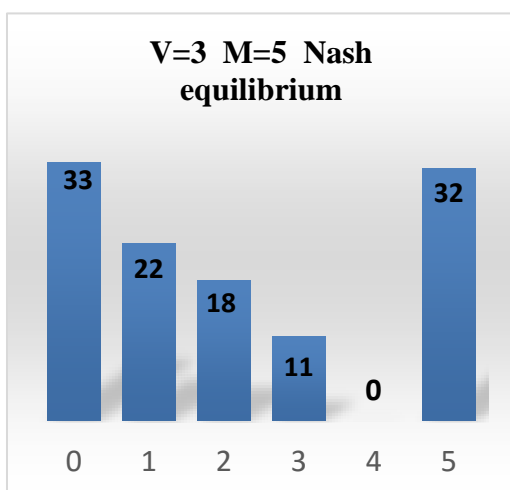**Students' way to play**

bid	0	1	2	3	4	5
count	44	10	2	24	19	17

Figure 4a

V=30 M=60 (bids from tens to tens)
Students' way to play

bid	0	10	20	30	40	50	60
count	36	6	3	23	5	6	30

Figure 4b

V=3 M=5 Nash equilibrium

bid	0	1	2	3	4	5
count	33	22	18	11	0	32

Figure 2a (with number of students)

V=30 M=60 Nash equilibrium

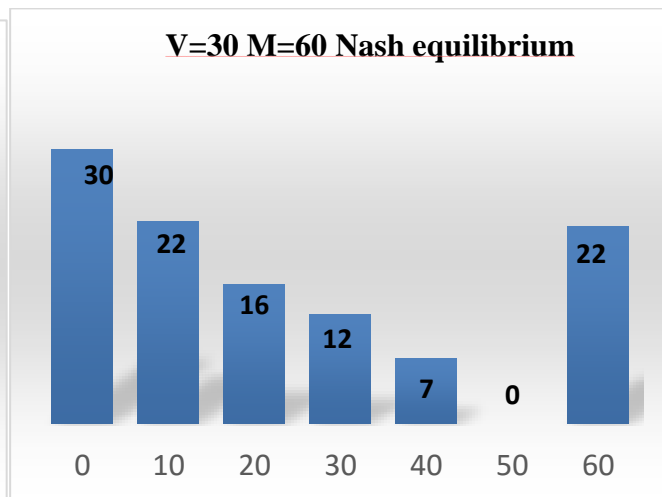| bid	0	10	20	30	40	50	60
count	30	22	16	12	7	0	22

Figure 2b (with number of students)

The students' distributions do not fit with the mixed NE distributions. The main difference concerns low bids different from 0. The probabilities assigned to low bids (1 and 2 in game 1, 10 and 20 in the second game) by the NE are much higher than the frequencies with which the students play these bids. Whereas bids 1 and 2, in the first played game, are played with probability 34.9% in the NE, they are only played with probability 10.5% by the students. And whereas the low bids 10 and 20, in the second game, are played with probability 34.55% in the NE, they are only played with probability 8.3% by the students.

Another strong difference is about the probability assigned to the value V of the prize (3 in the first game, 30 in the second game). Students bid this value much more often than in the NE (20.5% versus 9.2% in the first game, 21.1% versus 11.1% in the second game).

We can also observe that the students more often bid 0 than in the NE (38% versus 28.3% in the first game, 33% versus 27.7% in the second game), though the difference in the probabilities in the second game may not be statistically significant.

The way students play bids higher than V is different in both games. Whereas 31% of them play both bids 4 and 5 almost with the same probabilities in the first game (in contrast to the

NE that assigns probability 0 to bid 4 and 27.6% to bid 5), students, like the NE, assign a small probability to bids 40 and 50 (10.1% for the students, 6.7% in the NE) in the second game, and a large probability to bid 60 (27.5% for the students, 20% in the NE).

By putting these observations together, it is obvious that the students' probabilities are different from the Nash distribution, and - this matters more- that the shapes of the students' distributions are quite different from the mixed NE one.

Well, in our opinion, these differences simply highlight that the philosophy of a mixed NE doesn't fit with the way to play of real players. Let us justify our point of view by turning to best reply matching.

4. Philosophy of mixed Nash equilibria and philosophy of best reply matching.

We first recall Kosfeld et al.'s(2002) Best Reply Matching (BRM) equilibrium.

Definition 1 (Kosfeld & al. 2002): Normal form Best Reply Matching equilibrium

Let $G=(\mathcal{N}, S_i, \succ_i, i \in \mathcal{N})$ be a game in normal form($\mathcal{N}$ is the set of players, S_i is player i's set of pure strategies). A mixed strategy p is a BRM equilibrium if for every player $i \in \mathcal{N}$ and for every pure strategy $s_i \in S_i$:

$$p_i(s_i)= \sum_{s_{-i} \in B_i^{-1}(s_i)} \frac{1}{Card\ B_i(s_{-i})} p_{-i}(s_{-i})$$

where $B_i(s_{-i})$ is the set of player i's best replies to the strategies s_{-i} played by the other agents.

In a BRM equilibrium, the probability assigned to a pure strategy by player i is linked to the probability assigned to the opponents' strategies to which this pure strategy is a best reply: if player i's opponents play s_{-i} with probability $p_{-i}(s_{-i})$, and if the set of player i's best responses to s_{-i} is the subset of pure strategies $B_i(s_{-i})$, then each strategy of this subset is played with the probability $p_{-i}(s_{-i})$ divided by the cardinal of $B_i(s_{-i})$.

This criterion builds on the notion of rationalizability developed by Bernheim (1984) and Pearce (1984), according to which a strategy s_i is rationalizable if it is a best response to at least one profile s_{-i} played by the other players. Kosfeld et al. (2002) observe that, if the opponents often play s_{-i}, then s_i often becomes the best response, so should often be played, for actions and responses to be consistent. In other words, if, for example A_1 –respectively B_1- is player 1's best response to player 2's action B_2 –respectively A_2-, and A_2 –respectively B_2- is player 2's best response to player 1's action B_1 –respectively A_1-, then a BRM equilibrium can lead player 1 to play A_1 and B_1 with probabilities 1/3 and 2/3, and player 2 to play A_2 and B_2 with probability 2/3 and 1/3. So player 1 plays A_1 as often as player 2 plays the action B_2 to which A_1 is the best response, and she plays B_1 as often as player 2 plays the action A_2 to which B_1 is the best response. And vice versa for player 2. Kosfeld et al., in some way, rationalize the probabilities of a player by the other players' probabilities. And this kind of behavior is far from a mixed NE behavior.

Let us illustrate the concepts on the 2 by 2 normal form game in matrix 3.

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                    Player 2
                    A₂          B₂
Player 1   A₁   ( 2.9 , 1)  ( 5 , 5 )
           B₁   ( 3 , 3 )   ( 1 , 2.9)
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Matrix 3

We call p and 1-p, respectively q and 1-q, the probabilities assigned to A_1 and B_1 – respectively to A_2 and B_2. The mixed NE leads to equalize the payoffs obtained with A_1 and B_1, otherwise player 1 would only play the action leading to the highest payoff. So we get $2.9q+5(1-q)= 3q+1-q$, i.e. $q=40/41$, i.e. we get a condition on player 2's probabilities. We also have to equalize the payoffs obtained by player 2 with A_2 and B_2, $p+3(1-p)= 5p+2.9(1-p)$, so we get $p=1/41$, a condition on player 1's probabilities. This may seem quite strange. First, a player's probabilities have no impact on his own payoff. They only ensure that the opponent is indifferent between his actions in the NE support. So, when player 1 plays A_1 with probability 1/41 and B_1 with probability 40/41, these probabilities mean nothing for herself. She could play A_1 and B_1 each with probability ½, given that, due to player 2's probabilities, she is indifferent between A_1 and B_1. Second, if she chooses the probabilities 1/41 and 40/41, it is only to help player 2 to become indifferent between his two actions. This namely explains why she plays B_1 with a probability close to 1, despite B_1 is not interesting for her when considering the range of payoffs (2.9 and 5 for A_1, 3 and 1 for B_1). As a matter of facts, she plays B_1 with a probability close to 1 and A_1 with a probability close to 0 because player 2's payoffs- when he plays A_2 and B_2- are close when she plays B_1 (he gets 3 with A_2 and 2.9 with B_2) whereas they are very different when she plays A_1 (he gets 1 with A_2 and 5 with B_2). And vice versa for player 2.

Well, we simply think that real players do not choose probabilities in this way. In real life, when somebody plays A with probability 1/41 and B with probability 40/41, it is because she thinks that B is much more often her best response than A, 40 times more often, which justifies that she plays B with probability 40/41. In real life, probabilities –at least often- simply translate the frequency with which an action is seen to be a best response, and, as a consequence, the frequency with which we are ready to play it. And this is what is done in the BRM equilibrium. Let us define it for the game in matrix 3:

A_1 is player 1's best response to B_2, so has to be played as often as B_2, i.e. $p= 1-q$

B_1 is player 1's best response to A_2, so has to be played as often as A_2, i.e. $1-p= q$

A_2 is player 2's best response to B_1, so has to be played as often as B_1, i.e. $q= 1-p$

B_2 is player 2's best response to A_1, so has to be played as often as A_1, i.e. $1-q= p$

And $0≤p≤1$, $0≤q≤1$. So, for the studied game, we get an infinite number of BRM equilibria, characterized by the fact that player 1 plays A as often as player 2 plays B, and plays B as often as player 2 plays A.

Let us make 3 observations:

- First, the BRM way to define probabilities allows to cope with the asymmetric pure strategy NE. Given that A_1 is the best response to B_2 and B_2 is the best response to A_1, the BRM equilibrium allows player 1 to play A_1 with probability 1 and player 2 to play B_2 with probability 1, given that A_1 is player 1's best response as often as player 2 plays B_2 and vice versa. So, in this game, the pure strategy NE are also BRM equilibria.

- Second, it is important to note that the mean payoff in the mixed NE (here 121/41) may be higher or lower than the mean payoff in the BRM equilibrium. In the studied game, as long as we choose p between 1/41 and 20/41, the players get more with the NE than with the BRM equilibrium, but for p<1/41 and p>20/41 the players get more with the BRM equilibrium. What is different is the philosophy of the payoffs. In the mixed NE, each player wants to get the same payoff with each of the played strategies (i.e. his strategies in the support of the mixed NE). This is not convincing (even if mathematically logical). Especially if the support of the mixed NE is the whole set of pure strategies, have you ever seen a player who says: "let's try to put probabilities on my strategies so that the opponents get the same payoff with all their pure strategies" ?
  Real behavior is less sophisticated (and less strange): players simply try to behave at best. In the BRM equilibrium, each played strategy is the best one as regards at least one strategy played by the opponent. In some degree, people, despite they assign probabilities to strategies, still behave in a pure strategy way. They choose an action with a high probability if it is the best response to other actions, also chosen with a high probability. They simply try to be consistent with the way other players play, adapting their probability to play an action to the probabilities with which the others play the strategies to which this action is a best reply. This way to deal with probabilities has no link with the mixed NE way to deal with probabilities.
- Third, let us observe that, in the studied game, the mixed NE is also a BRM equilibrium (because p=1-q=1/41) - most often mixed NE are not BRM equilibria. Yet the justification of this special BRM is not the mixed NE one. Whereas player 1 and player 2, in the NE, calculate the probabilities by equalizing the payoffs obtained with A and B for both players, in the BRM, player 1 plays A with probability 1/41 because it is the best response to B_2 which is also played with probability 1/41 and she plays B_1 with probability 40/41, because it is the best reply to A_2, which is played with the same probability 40/41 (and the symmetric explanation holds for player 2). So both actions are played because each is a best response, and not because they lead to the same payoff.

Let us add a generalization of the BRM concept. When there are several best replies to a profile s_{-i}, we think that there is no reason to demand that each best reply is played with the same probability, so we think that is reasonable to generalize Kosfeld et al.'s criterion by allowing to play the different best replies with different probabilities as follows:

Definition 2 Generalized BRM equilibrium in normal form games (Umbhauer 2007)

Let $G=(\mathcal{N}, S_i, \succ_i \ i \in \mathcal{N})$ be a game in normal form. A mixed strategy p is a Generalized BRM (GBRM) equilibrium if for every player $i \in \mathcal{N}$ and for every pure strategy $s_i \in S_i$, :

$$p_i(s_i) = \sum_{s_{-i} \in B_i^{-1}(s_i)} \delta_{s_i} \, p_{-i}(s_{-i})$$

with $\delta_{s_i} \in [0, 1]$ for any s_i belonging to $B_i(s_{-i})$ and $\sum_{s_i \in B_i(s_{-i})} \delta_{s_i} = 1$.

Pure Nash equilibria, by contrast to mixed ones, are automatically GBRM equilibria (out of Umbhauer 2016): if player 1 plays A and player 2 plays B in a pure strategy Nash equilibrium –so they play the actions with probability 1-, player 1 plays A as often as the opponent plays

the action to which A is a best reply, i.e. B, and player 2 plays B as often as player 1 plays the action A to which B is the best reply.

5. Best reply matching and generalized best reply matching equilibria in second price all-pay auctions, links with the real way to play and focal points

We look for the BRM equilibria in both games studied by the students. We first work with the first game[4]. To do so, we write the best reply matrix 4a where b_i means that player i's action is a best reply to the opponent's action, i=1,2.

Matrix4a (Player 2 across top, Player 1 down side)

1		q_0 / 0	q_1 / 1	q_2 / 2	q_3 / 3	q_4 / 4	q_5 / 5
p_0	0		b_2	b_2	b_1b_2	b_1b_2	b_1b_2
p_1	1	b_1		b_2	b_2	b_2	b_2
p_2	2	b_1	b_1		***b_2***	b_2	b_2
p_3	3	b_1b_2	b_1	b_1		b_2	b_2
p_4	4	b_1b_2	***b_1***	b_1	b_1		
p_5	5	b_1b_2	b_1	b_1	b_1		

Matrix 4b (Player 2 across top, Player 1 down side)

1		q_0 / 0	q_{10} / 10	q_{20} / 20	q_{30} / 30	q_{40} / 40	q_{50} / 50	q_{60} / 60
p_0	0		b_2	b_2	b_1b_2	b_1b_2	b_1b_2	b_1b_2
p_{10}	10	b_1		b_2	b_2	b_2	b_2	b_2
p_{20}	20	b_1	b_1		b_2	b_2	b_2	b_2
p_{30}	30	b_1b_2	b_1	b_1		b_2	b_2	b_2
p_{40}	40	b_1b_2	b_1	b_1	b_1			
p_{50}	50	b_1b_2	b_1	b_1	b_1			
p_{60}	60	b_1b_2	b_1	b_1	b_1			

For example, the bold b_1 in italics means that bid 4 is one of player 1's best replies to player 2's bid 1, and the bold b_2 in italics means that bid 3 is one of player 2's best replies to player 1's bid 2. We write p_i, respectively q_i, the probability assigned to bid i by player 1, respectively by player 2, i from 0 to 5.   So we get the system of equations:

$p_0 = q_3/3 + q_4 + q_5$ $\qquad$ $q_0 = p_3/3 + p_4 + p_5$

$p_1 = q_0/5$ $\qquad$ $q_1 = p_0/5$

$p_2 = q_0/5 + q_1/4$ $\qquad$ $q_2 = p_0/5 + p_1/4$

$p_3 = q_0/5 + q_1/4 + q_2/3$ $\qquad$ $q_3 = p_0/5 + p_1/4 + p_2/3$

$p_4 = q_0/5 + q_1/4 + q_2/3 + q_3/3$ $\qquad$ $q_4 = p_0/5 + p_1/4 + p_2/3 + p_3/3$

$p_5 = q_0/5 + q_1/4 + q_2/3 + q_3/3 = p_4$ $\qquad$ $q_5 = p_0/5 + p_1/4 + p_2/3 + p_3/3 = q_4$

$p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = 1$ $\qquad$ $q_0 + q_1 + q_2 + q_3 + q_4 + q_5 = 1$

The solution of this system of equations is :

$p_0 = q_0 = 180/481 = 37.4\%$, $p_1 = q_1 = p_0/5 = 7.5\%$, $p_2 = q_2 = p_0/4 = 9.4\%$, $p_3 = q_3 = p_0/3 = 12.5\%$, $p_4 = p_5 = q_4 = q_5 = 4p_0/9 = 16.6\%$

These probabilities, reproduced in table 2a are far from the Nash equilibrium ones and they fit much better with the students' probabilities, except p_2 (higher) and p_3 (lower). This proximity is due to the fact that BRM exploits main facts also observed by the students, namely that bids 1 and 2 are seldom best responses. As a matter of facts, bidding 1 and 2 seldom leads to win the prize (namely if the opponent bids 3, 4 or 5) and, if you don't win, you lose money with these bids (so it is better to bid 0). In fact, bid 1 is a best response only if the opponent bids 0 (and in this case, bids 2,3,4,5 are also best responses), bid 2 is a best response only if the

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[4] The results linked to the first game are partly out of Umbhauer 2016.

opponent bids 0 or 1 and in these two cases, bids 3, 4 and 5 are also best responses. By contrast, 0, 3, 4 and 5 are often best responses (bid 0 is the best response to bids 4 and 5 and one best response to bid 3, bid 3 is a best response to bids 0, 1 and 2, bids 4 and 5 are best responses to bids 0, 1, 2 and 3).

V=3, M=5 frequencies probabilities	Nash equilibrium	Students	BRM equilibrium
0	28.3%	38%	37.4%
1	19.5%	9%	7.5%
2	15.4%	1.5%	9.4%
3	9.2%	20.5%	12.5%
4	0	16%	16.6%
5	27.6%	15%	16.6%

Table 2a

V=30,M=60 frequencies probabilities	Nash equilibrium	Students	BRM equilibrium
0	27.7%	33%	39.15%
10	20.5%	5.5%	6.5%
20	14.1%	2.8%	7.8%
30	11.1%	21.1%	9.8%
40	6.7%	4.6%	12.25%
50	0%	5.5%	12.25%
60	20%	27.5%	12.25%

Table 2b

The BRM equilibrium, in the second game, whose best reply matrix is given in matrix 4b, is given by: $p_0=q_0=240/613=39.15\%$ $p_{10}=q_{10}=p_0/6 =40/613=6.5\%$, $p_{20}=q_{20}=p_0/5=48/613=7.8\%$, $p_{30}=q_{30}=p_0/4= 60/613= 9.8\%$, $p_{40}=p_{50}=q_{40}=q_{50}= p_{60}=q_{60}= 5p_0/16 = 75/613=12.25\%$
(given the system of equations:

$p_0= q_{30}/4 +q_{40}+q_{50}+q_{60}$ 　　　　　$q_0= p_{30}/4+p_{40}+p_{50}+p_{60}$
$p_{10}= q_0/6$ 　　　　　$q_{10}= p_0/6$
$p_{20}= q_0/6+q_{10}/5$ 　　　　　$q_{20}= p_0/6+p_{10}/5$
$p_{30}= q_0/6+q_{10}/5+q_{20}/4$ 　　　　　$q_{30}= p_0/6+p_{10}/5+p_{20}/4$
$p_{40}= q_0/6+q_{10}/5+q_{20}/4+q_{30}/4$ 　　　　　$q_{40}= p_0/6+p_{10}/5+p_{20}/4+p_{30}/4$
$p_{50}= q_0/6+q_{10}/5+q_{20}/4+q_{30}/4 = p_{40}$ 　　　　　$q_{50}= p_0/6+p_{10}/5+p_{20}/4+p_{30}/4 = q_{40}$
$p_{60}= q_0/6+q_{10}/5+q_{20}/4+q_{30}/4 = p_{40}$ 　　　　　$q_{60}= p_0/6+p_{10}/5+p_{20}/4+p_{30}/4 = q_{40}$
$p_0 +p_{10}+ p_{20}+p_{30}+p_{40}+p_{50}+p_{60}=1$ 　　　　　$q_0 +q_{10}+q_{20}+q_{30}+q_{40}+q_{50}+q_0=1$)

Well, this time, the BRM equilibrium is different from the NE and from the students' behavior. But this may be due to the fact that students, especially when the number of bids grows and when there are several best replies, do not play all best replies with the same probability, and may even choose to play only some of them, as allowed by the GBRM equilibrium. Focal bids may be preferred each time they are best replies. So, observe that bids 40, 50 and 60 are best replies to the bids 0, 10, 20 and 30, which explains that they are each played with the same probability 12.5% in the BRM equilibrium. With the GBRM concept, a player can choose to play more often some best responses than others, provided that the sum of probabilities is the same, i.e. 12.5x3 = 37.5%. And we can observe that the sum of probabilities assigned to 40, 50 and 60 by the students is 37.6%.

To develop this point, let us suppose that students more focus on threshold values, i.e., in this game, 30 (the value of the prize), 60 (the maximal bid and budget) and 0 (the cautious bid). So suppose that, each time the best responses include one or several of these bids, the players only play these bids. For example, when player 1 bids 10, player 2 only best replies with bid 30 and bid 60, despite bids 20, 40 and 50 are also best responses. This leads to the GBRM

14

matrix 5 (consider only the b_1 and b_2 (underlined and not underlined), the B_1 and B_2 are used in a further study).

Player 2

Player1		q_0 0	q_{10} 10	q_{20} 20	q_{30} 30	q_{40} 40	q_{50} 50	q_{60} 60
p_0	0		B_2		b_1b_2	b_1	b_1	b_1b_2
p_{10}	10	B_1		B_2	b_2			b_2
p_{20}	20		B_1		b_2			b_2
p_{30}	30	b_1b_2	b_1	b_1		B_2	B_2	b_2
p_{40}	40	b_2			B_1			
p_{50}	50	b_2			B_1			
p_{60}	60	b_1b_2	b_1	b_1	b_1			

Matrix 5

The set of equations becomes:

$p_0= q_{30}/2 +q_{40}+q_{50}+q_{60}$　　　　$q_0= p_{30}/2+p_{40}+p_{50}+p_{60}$

$p_{10}= 0$　　　　　　$q_{10}= 0$

$p_{20}= 0$　　　　　　$q_{20}= 0$

$p_{30}= q_0/2+q_{10}/2+q_{20}/2$　　　$q_{30}= p_0/2+p_{10}/2+p_{20}/2$

$p_{40}= 0$　　　　　　$q_{40}= 0$

$p_{50}= 0$　　　　　　$q_{50}= 0$

$p_{60}= q_0/2+q_{10}/2+q_{20}/2+q_{30}/2$　　　$q_{60}= p_0/2+p_{10}/2+p_{20}/2+p_{30}/2$

$p_0 +p_{10}+ p_{20}+p_{30}+p_{40}+p_{50}+p_{60}=1$ $q_0 +q_{10}+q_{20}+q_{30}+q_{40}+q_{50}+q_0=1$

which simplifies to :

$p_0= q_{30}/2 +q_{60}$　　　$q_0= p_{30}/2 +p_{60}$

$p_{30}= q_0/2$　　　　　$q_{30}= p_0/2$

$p_{60}=q_0/2+ q_{30}/2$　　　$q_{60}= p_0/2+p_{30}/2$

$p_0+p_{30}+p_{60}=1$　　　$q_0+q_{30}+q_{60} =1$

whose　　solution　　is　　$p_0=q_0=4/9=44.4\%$　　$p_{30}=q_{30}=p_0/2=22.2\%$,　　$p_{60}=q_{60}=3p_0/4=33.3\%$ $p_{10}=q_{10}=p_{20}=q_{20}= p_{40}=q_{40}=p_{50}=q_{50}= 0$.

So we get a 3 peak distribution which is similar to the students' one as regards the shape (highest peak on 0, second highest peak on 60 and lowest peak on 30) (see Figure 5a which gives the probabilities in number of students and table 3).

We can even get closer to the students' probabilities, by not completely excluding 10, 20, 40 and 50 from the played best responses. So we may suppose that players, when selecting best responses, focus on 0, 30 and 60 but also play 10 as the closest best response to 0, 20 as the closest best response to 10, and 40 and 50 as best responses to 30 (we add the B_1 and B_2 in matrix 5). So we get the system of equations

$p_0= q_{30}/4 +q_{40}+q_{50}+q_{60}$　　　　$q_0= p_{30}/4+p_{40}+p_{50}+p_{60}$

$p_{10}= q_0/3$　　　　　　$q_{10}= p_0/3$

$p_{20}= q_{10}/3$　　　　　$q_{20}= p_{10}/3$

$p_{30}= q_0/3+q_{10}/3+q_{20}/2$　　　$q_{30}= p_0/3+p_{10}/3+p_{20}/2$

$p_{40}= q_{30}/4= p_{50}$　　　　$q_{40}= p_{30}/4=q_{50}$

$p_{60}= q_0/3+q_{10}/3+q_{20}/2+q_{30}/4$　　　$q_{60}= p_0/3+p_{10}/3+p_{20}/2+p_{30}/4$

$p_0 +p_{10}+ p_{20}+p_{30}+p_{40}+p_{50}+p_{60}=1$　　　$q_0 +q_{10}+q_{20}+q_{30}+q_{40}+q_{50}+q_0=1$)

whose solution is : $p_0=q_0=72/203=35.5\%$, $p_1=q_1=24/203=11.8\%$, $p_2=q_2=8/203=3.9\%$, $p_3=q_3=36/203=17.7\%$, $p_4=p_5=q_4=q_5=9/203=4.45\%$ and $p_6=q_6=45/203=22.2\%$.

V=30,M=60 bid	Nash equilibrium	Students	GBRM equilibrium with bids 0, 30 and 60	GBRM equilibrium, with special focus on 0, 30 and 60
0	27.7%	33%	44.5%	35.5%
10	20.5%	5.5%	0	11.8%
20	14.1%	2.8%	0	3.9%
30	11.1%	21.1%	22.2%	17.7%
40	6.7%	4.6%	0	4.45%
50	0%	5.5%	0	4.45%
60	20%	27.5%	33.3%	22.2%

Table3
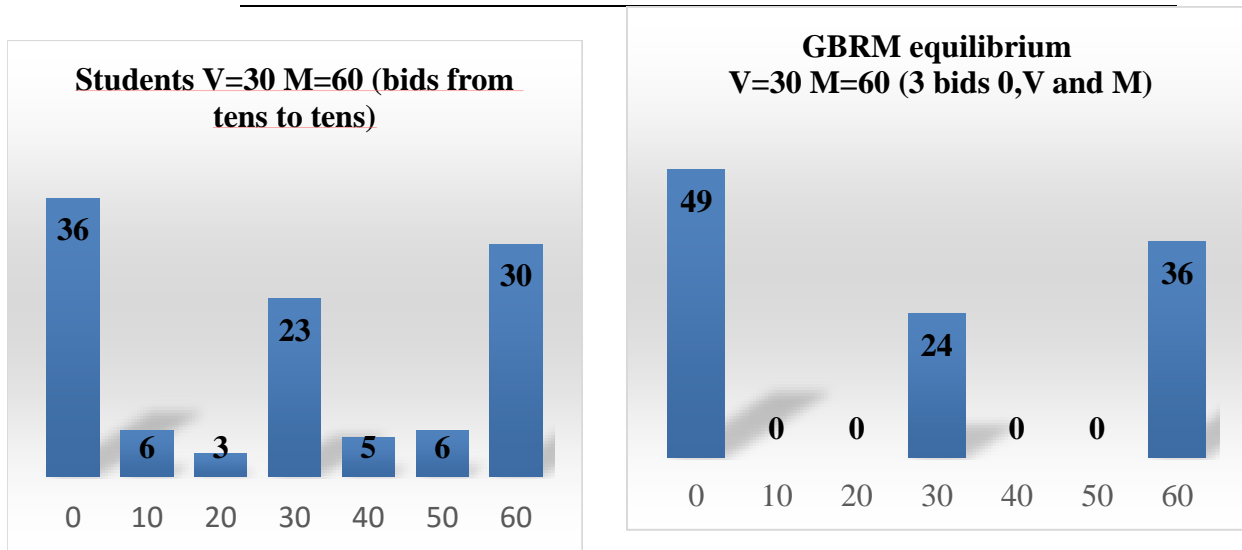
Figure 4b

Figure 5a

Figure 5b

16
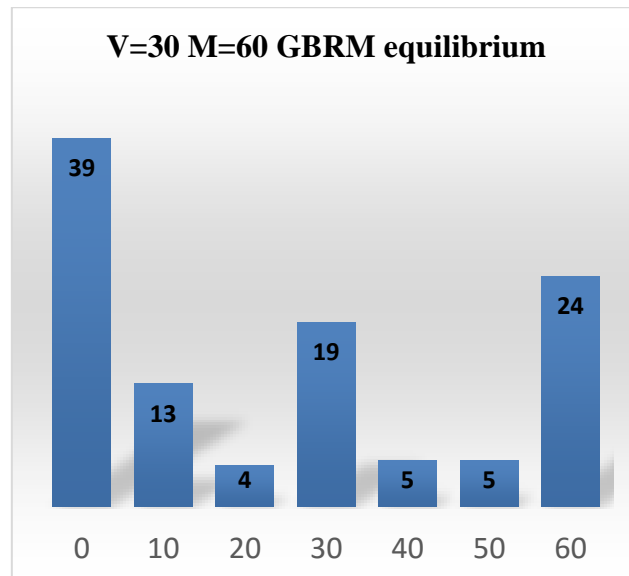
So, without extravagant assumptions, we can get close to the students' distribution (both in probabilities (see table 3) and in shape (see Figure 5b which gives the probabilities in number of students).

But let us come back to the extreme case where students only focus on 0, 30 and 60 (observe that more than 4/5 of the students only play these bids) and let us reduce the game to these 3 bids (we need no other bid to have always a best response in the original game), to see how the mixed NE and BRM proceed in this reduced game. So we get the new normal form game in matrices 6a (V=30, M=60) and 6b (any values V and M). For both matrices 6a and 6b, the best reply matrix is matrix 6c.

		Player 2		
		0	30	60
	0	(75,75)	(60, 90)	(60, 90)
Player1	30	(90,60)	(45,45)	(30, 60)
	60	(90,60)	(60, 30)	(15,15)

Matrix 6a

		Player 2		
		0	V	M
	0	(M+V/2, M+V/2)	(M, M+V)	(M, M+V)
	V	(M+V, M)	(M-V/2, M-V/2)	(M-V, M)
	M	(M+V, M)	(M, M-V)	(V/2,V/2)

Matrix 6b

		Player 2		
		0	V	M
	0		$b_1 b_2$	$b_1 b_2$
Player1	V	$b_1 b_2$		b_2
	M	$b_1 b_2$	b_1	

Matrix 6c

Regardless of V and M, the BRM equilibrium equations, which are GBRM equations in the original game, become:

$p_0= q_V/2 +q_M$ $\qquad$ $q_0= p_V/2 +p_M$
$p_V= q_0/2$ $\qquad$ $q_V= p_0/2$
$p_M=q_0/2+ q_V/2$ $\qquad$ $q_M= p_0/2+p_V/2$
$p_0+p_V+p_M=1$ $\qquad$ $q_0+q_V+q_M =1$

It follows $p_0=q_0=4/9$ $p_V=q_V=2/9$, $p_M=q_M=3/9$.

These probabilities exploit true facts highlighted in the best reply matrix 6c. Bid 0 is more played than M because bid 0 is the unique best response to M (so it will be played at least as much as M) but also one among the 2 best responses to V (0 and M are best responses), which explains the higher probability on bid 0. M is more played than V because V is only a best response to bid 0, yet M is also a best response to bid 0 (so M will be played at least as much as V), and it is in addition one among the 2 best responses to V (this explains that M will be played with a higher probability than V). So we easily get the hierarchy of probabilities $p_0>p_M>p_V$. And a real player may observe that bid 0 is the only best answer to M –so he may spontaneously bid 0 at least as often as he expects that M is played by the opponent- ; he may also spontaneously observe that V is less often a best response than M which may lead him to

play V less often than M. So we can expect that real players indeed play in accordance with the hierarchy $p_0>p_M>p_V$.

Yet, let us first observe that such a behaviour may be dangerous in terms of payoffs. In contrast to the mixed NE concept, the BRM equilibrium is an ordinal concept. It takes into account differences in payoffs but not the value of the payoffs. For example, in matrix 6a, the low payoff (15) gotten by a player who bids 60 when meeting another player who bids 60 never appears in the equations. More generally, the values $p_0=q_0=4/9$ $p_3=q_3=2/9$ $p_6=q_6=3/9$ only derive from the comparisons M+V>M+V/2, M>M-V/2, M>V/2 and M>M-V, which are true regardless of the values V, M and the ratio M/V (provided M>V/2 which is a common assumption). This not typical to BRM given that other criteria (like risk dominance or, more simply, the pure strategy Nash equilibrium for example) also only focus on differences in payoffs. By contrast, the mixed NE is a cardinal concept, given that it equalizes the payoffs obtained with the bids in the equilibrium support. So the mixed NE procedure is less intuitive but it is more save in terms of payoffs.

So the mixed NE in the game in matrix 6a leads to bid 0 with probability 0.6, to bid 30 with probability 0.2 and to bid 60 with the same probability 0.2, which leads to the same mean payoff 69 for each of the three bids, whereas the BRM equilibrium leads only to the mean payoff 62.4. In this game, the mixed NE may even become intuitive when focusing on payoffs. So the same probability on 30 and 60 can be explained as follows: what you lose in front of somebody bidding 30 when you bid 30 instead of 60 (60-45) is equal to what you gain in front of somebody bidding 60 when you bid 30 instead of 60 (30-15). And you get the same payoff with both bids 30 and 60 in front of somebody who bids 0 (90=90). The higher probability assigned to bid 0 by contrast is mainly due to the fact that putting a high probability on bid 0 better allows the payoffs to become equal (because 90-75=15=60-45 <60-15), which is less intuitive.

To precise this remark on payoffs, we summarize below in table 4 the payoffs obtained with the BRM equilibrium and the mixed NE in the game in matrix 6b (we always suppose M>V/2).

	BRM probabilities	NE probabilities M>3V/2	NE probabilities V/2<M<3V/2
0	4/9	(2M-V)/(4M-3V) $\rightarrow$ ½ when M$\rightarrow\infty$	(2M-V)/2M
V	2/9	(2M-3V)(4M-3V) $\rightarrow$ ½ when M$\rightarrow\infty$	0
M	3/9	V/(4M-3V) $\rightarrow$ 0 when M$\rightarrow\infty$	V/2M
Mean payoff	8M/9+24.5V/81	$(4M^2-2VM-0.5V^2)/$ (4M-3V)	$(2M^2+VM-0.5V^2)/$ 2M
Net payoff = Mean Payoff -M	24.5V/81-M/9	V(M-0.5V)/(4M-3V)	V(M-0.5V)/2M

Table 4

It follows that a player never loses money in a mixed NE (which is logical because bid 0 leads to no loss). It also follows that, for M>3V/2 the mean payoff is always higher in the mixed NE than in the BRM equilibrium, and that the BRM equilibrium mean payoff can be higher than the mixed NE one only if M <0.85V. Contrary to the mixed NE, the BRM equilibrium

can lead to a negative net payoff when $M>24.5V/9$ (this can't happen with the Nash equilibrium given that the probability on M goes to 0 when M becomes large, V being constant).

Yet this doesn't mean that GBRM in the original game is necessarily dangerous. Up to now, we focused on the values 0, V and M just to get close to our students in the second experiment, who seemed to prefer 0, V and M as best responses. It is important to note that our students did not lose money (mean payoff = 62.4 > 60), namely because M/V was not too large: in the experiment, $M=2V<24.5V/9$. So it is possible that the students focused on 0, $V=30$ and $M=60$, because they estimated that the possible loss with M was not large enough to prevent them from bidding M. It may be that the students would have behaved differently for other values of V, M, and the ratio M/V. It may be that there exists a kind of bifurcation in the focal bids chosen as best replies when the values of M, V and M/V change and exceed threshold values. For example, let us suppose, as allowed by GBRM, that, if M/V becomes large, players, in the original game, only best respond with 0 and V; this is possible because either 0 or V belong to the best responses to each possible bid (0 is a best reply to all bids higher or equal to V, and V is a best response to each bid lower than V), as illustrated in matrix 5 (see the underlined $\underline{b_1}$ and $\underline{b_2}$). In that case, the system of GBRM equations, after deleting all the null probabilities, reduces to $q_0= p_V$ and $q_V=p_0$, each player bidding 0 as often as the opponent bids V and vice versa. It is interesting to note that the symmetric GBRM equilibrium behaviour, which consists to bid 0 and V with probability ½, leads to a net payoff V/4, which is never negative and becomes larger than the mixed NE payoff when V becomes large.

To summarize, if real players, as the students in our experiments, behave in accordance with GBRM, that allows to combine BRM and a focus on some focal actions, then it becomes crucial to know if players sufficiently promptly bifurcate in their choice of best replies to avoid losing money. For example, they do not lose money by focusing on 0, V and M as best responses as long as $M/V <24.5/9$, but they lose money if $M/V > 24.5/9$. So, if they switch from a game where M/V=2 to a game where M/V=4, to avoid losing money they can switch from the focal values 0, V and M, to the focal values 0 and V. But we know that a change in focal values requires time; so players may lose money before changing these values. Another way to cope with our result is to say that second price all-pay auction games should be regulated. If they were casino games, some Regulatory Authority like ARJEL in France could set an upper bound for M/V, to avoid that players lose a lot of money. M=2V or even M=V seem to be natural limits.

In the next section we come back to the BRM equilibrium, so we suppose that each best reply is played with the same probability. If so, for large values of M, people may lose $1/12^{th}$ of their budget.

6. Best reply matching equilibrium for any V and M, how one can lose a lot of money

Proposition 1 gives the symmetric BRM equilibrium in the second price all-pay auction game for any integer values V and M, M≥V.

> **Proposition 1**
> *The symmetric BRM equilibrium is given by:*
> $q_0 = 1/(2 + \sum_{i=0}^{V-1} [\frac{1}{M-i}] - [1/(M-V+1)^2])$
> $q_i = q_0/(M-i+1)$,   *i from 1 to V*
> $q_i = q_0 (M-V+2)/(M-V+1)^2$,   *i from V+1 to M*
> *It follows that:* $\sum_{i=V+1}^{M} q_i < q_0 < \sum_{i=V}^{M} q_i$

Proof see Appendix 3

Let us comment on this equilibrium. Let us first make 5 remarks.

The first is about the shape of the BRM equilibrium distribution and the mixed NE distribution.

In the BRM equilibrium, q_i is increasing in i for i from 1 to V+1, and is constant from V+1 to M, a result in sharp contrast with the mixed NE probabilities that are decreasing from 1 to M-V/2 and null from M-V/2 to M (excluded). BRM clearly takes into account that a higher bid is more often a best reply than a lower one (different from 0), in that each bid b (different from 0) is a best reply to all the bids lower than b, if b≤V+1, and a best reply to all bids from 0 to V if b is higher than V. And bid 0, in contrast to the other low bids, has a special status in that it is a best reply to all bids from V to M.

Clearly, the Nash and the BRM distributions have no common points, except the fact that q_0 is higher than all the probabilities q_i, i from 1 to V, in both the BRM equilibrium and in the NE. The strong differences and the few similarities are highlighted in Figures 6 and 3a, which give the BRM equilibrium and the mixed NE for V=9 and M=12.
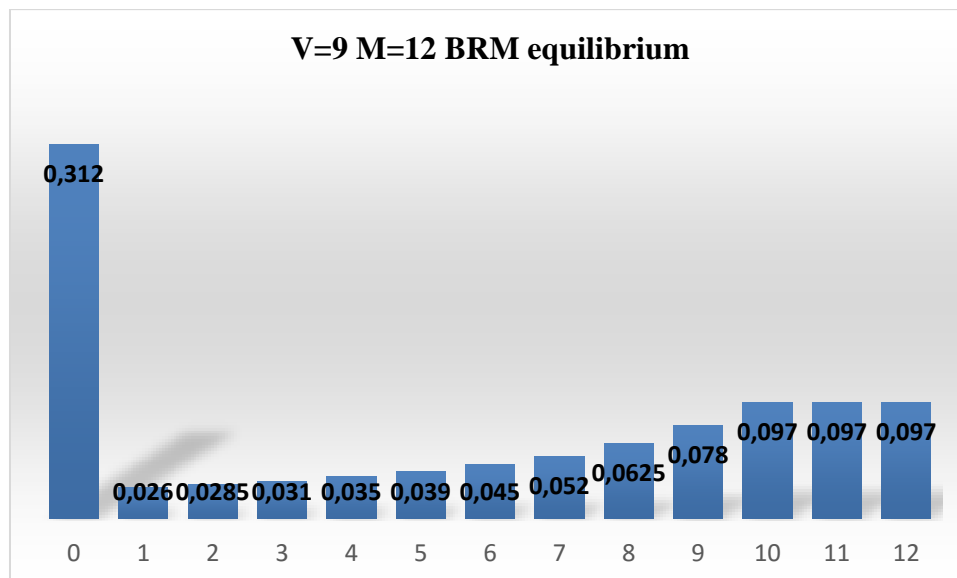

Figure 6

V=9 M=12 Nash equilibrium

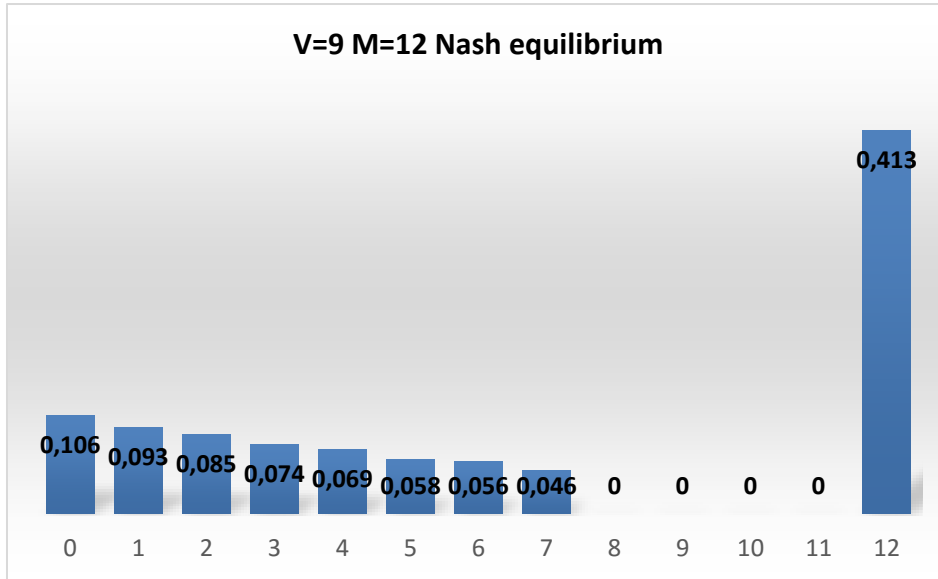| 0	1	2	3	4	5	6	7	8	9	10	11	12
0,106	0,093	0,085	0,074	0,069	0,058	0,056	0,046	0	0	0	0	0,413

Figure 3a

Second, let us talk about payoffs. As regards the payoffs in the BRM equilibrium and in the mixed NE, for M not too far from V, the BRM equilibrium payoffs may be higher than the NE ones. For example, for V=3 and M=5, the Nash equilibrium payoff is $1589.5/293 = 5.42$, whereas the BRM payoff is 5.46
$(= 180 \ (6.5\text{x}180+5(481\text{-}180)) + 36(8\text{x}180+5.5\text{x}36+4(481\text{-}180\text{-}36)) + 45(8\text{x}180+7\text{x}36+ 45\text{x}4.5+3\text{x}60+3\text{x}80+3\text{X}80) + 60(8\text{x}180+7\text{x}36+6\text{X}45+3.5\text{x}60+2\text{x}80+2\text{x}80) + 80(8\text{x}180 +7\text{x}36+6\text{X}45 + 5\text{x}60 +2.5\text{x}80+1\text{x}80) + 80(8\text{x}180+7\text{x}36+6\text{X}45 +5\text{x}60+4\text{x}80+1.5\text{x}80) \ /480^2 = 1262620.5/481^2 = 5.46)$
For V=30 and M=60 (and bids from tens to tens) the results are reversed: the Nash equilibrium payoff is 64.157 and the BRM payoff is slightly lower $(24082745/613^2 = 64.089)$.

Third, looking at Figures 6 and 3a highlights that it is difficult to speak about overbidding or underbidding, when comparing BRM behaviour and Nash behaviour, because the structure of behaviours is completely different. The only thing we can say is that, for M>2V, bids higher than the value of the prize focus more weight than the bids from 1 to V. As a matter of fact, given that $q_i = q_V + q_V/(M\text{-}V+1)$, for i from V+1 to M , we get
$\sum_{i=1}^{V} q_i < Vq_V < \sum_{i=V+1}^{M} q_i = (M\text{-}V)q_V + (M\text{-}V)q_0/(M\text{-}V+1)^2$
because $(M\text{-}2V)q_V + (M\text{-}V)q_0/(M\text{-}V+1)^2 > 0$

Fourth, one can observe that weak dominance has no impact in the BRM equilibrium, in that all the messages from V+1 to M are best responses to the same bids (from 0 to V). This isn't shocking given that the pure strategy Nash equilibria of the game also lead one player to bid 0 and the other player to bid any bid in [V, M].

Fifth, let us draw attention to the weight assigned to bid 0. Due to the special status of bid 0 - bid 0 is the unique best reply to all the bids from V+1 to M, and (only) one best reply to V -, it immediately follows that that $\sum_{i=V+1}^{M} q_i < q_0 < \sum_{i=V}^{M} q_i$. So, at least for M>2V, less than 1/3 of the probability is assigned to the bids from 1 to V, and the remaining probability is almost equally shared among bid 0 on the one hand, and the bids from V+1 to M on the other

hand. So there is a kind of atom on 0 (even if the game is discrete), which is not the case in the mixed NE. This remark generalizes as follows:

Proposition 2
The BRM equilibrium, for M≥V, assigns to bid 0 a higher probability than the mixed NE, at least for large values of V.

Proof see Appendix 4

Let us now give some additional results for two special values of the ratio M/V, M=2V and M=V, in that, from a regulation point of view, M=2V and M=V may appear as natural limits.

Proposition 3
For M=2V and V large, $q_0 \simeq 1/(2+ln(2))=0.371$, $\sum_{i=V}^{M} q_i \simeq 0,371$ and $\sum_{i=1}^{V-1} q_i \simeq 0.258$.

Proof see Appendix 5

Let us compare these values with the NE ones. In the mixed NE, which goes to the NE in the continuous game, we have:

$$\int_{V}^{M-V/2} \frac{e^{-\frac{x}{V}}}{V}\, dx + e^{1/2-M/V} = [-e^{-x/V}]_V^{3V/2} + e^{-3/2} = e^{-1} = 0.368$$

So we get almost the same weight on the messages from V to M, but not distributed in the same way. In the NE, M (=2V) is played with the probability $e^{-3/2}= 0.223$ whereas the weight (0.368-0.223=) 0.146 is shared among the messages from V to M-V/2= 3V/2 in a decreasing way, and the messages from 3V/2 to 2V (excluded) are played with probability 0.
And the remaining probability (1-0.368)= 0.632 is shared among the messages from 0 to V in a decreasing way but without atom on 0.

Proposition 4:
For M=V, $q_0=q_M \simeq 1/(1+ln(V)+\gamma)$ where γ is Euler's constant 0.577, and $q_i=1/[(1+ln(V)+\gamma)(V-i+1)]$, i from 1 to V

Proof see Appendix 5

Contrary to the case M=2V and V large, where bid 0 on the one hand, the set of bids from V to M on the other hand, were each played with a probability close to 0.371, we observe that, for M=V, bid 0 and the set of bids from V to M (which reduces to the bid V), are each played with the probability $1/(1+ln(V)+\gamma)$, which goes to 0 for V large. Yet this isn't strange, in that M=V is a very special case. If M=V, bid 0 is the best reply to only one bid, M (=V), so is played as often as M is played (that is why $q_M=q_0$). M is the bid that is most often the best reply (it is the best reply to all other bids), so it is played with the highest probability, but the other bids are also often best replies (each bid is the best reply to all bids lower than it); for example, M-1 is a best reply to all bids except M-1 and M. This explains that the other bids also focus high probabilities, which explains that 0 and M are played with a probability decreasing in V. Yet observe that this probability decreases slowly (for V= 200, M and 0 are

still each played with probability 0.145 (so the other 198 bids share the probability 0.71), for V=10000, 0 and M still focus 18.5% of the probability).

We now focus on the BRM equilibrium payoffs when M becomes large and V is a constant. So we worry about how much money players can lose in second price all-pay auction games when the limit budget is very large (M→∞) in comparison with V (so M/V →∞).

We observe that q_0 goes to ½, that all the other probabilities go to 0 (but still increasing in i from 1 to V+1 and constant from V+1 to M), that $\sum_{i=1}^{V} q_i < Vq_V = Vq_0/(M-V+1) \to 0$ and that $\sum_{i=V+1}^{M} q_i = q_0 (M-V+2)(M-V)/(M-V+1)^2 \to q_0$, i.e. goes to ½.

So, when M is large, the BRM probabilities are shared on bid 0 (probability ½) and homogenously shared on the set of bids from V+1 to M (probability ½ on this set). We have a kind of bimodal distribution, ½ on bid 0 and ½ on a set (each bid in the set being played with the same probability).

Observe that this result is close to the pure strategy Nash equilibrium spirit (for each player bidding 0, there is a player bidding i, with i higher than V), and quite far from the mixed NE (with no atom on any bid and decreasing probabilities on [0, +∞[(probability (1/V)db on 0)).

Yet this behaviour leads to a negative net payoff.

Proposition 5

For very large values of M, much larger than V (M→∞, V is a constant), the mean loss of a player, at the BRM equilibrium, is equal to $1/12^{th}$ of his budget.

For any value of M ≥V, the difference in payoffs between bid V+i and bid V+i+1 is equal to b(i-(M-2V)), i from 1 to M-V , where b is the probability to play each bid V+i, i from 1 to M-V. So

- for M≤2V, the payoff with bid V+i increases continuously with i, from i=1 to M-V.

- for M>2V, the payoff decreases with bid V+i , i from 1 to M-2V, and then increases.

For M large, bidding M-V leads to lose $1/4^{th}$ of the budget.

Proof see Appendix 6

So, for large values of M, a player may often suffer from the winner's curse. Playing high bids leads him to often win the prize, but he pays too much given that he often wins against players who bid too much, leading him to lose up to $1/4^{th}$ of his budget. And in average, he loses $1/12^{th}$ of his budget, which is a huge amount of money. Yet, if we consider the second price all-pay auction as a casino game, then proposition 5 draws attention to another fact. It is not sure that losing $1/12^{th}$ of the budget is sufficient to prevent a gambler from playing again, because he only loses $1/12^{th}$ of the budget! So, as regards pathological gambling, it may be that a player is incited to play and play again, despite the waste of money, which may become tragic. This leads us again to our previous remark on regulation. The above result is established for V constant and M going to infinity, i.e. for ratios M/V becoming large. Well, to avoid a huge waste of money, it may be necessary to limit the ratio M/V.

And let us make an additional observation that advocates for regulation too. The loss of $1/12^{th}$ of their budget by the two players is not lost for everybody. Somebody is getting this money,

i.e. 1/6th of M: the organizer of the auction game, who offers the prize. He gets M/6-V, which is a huge amount of money when M/V becomes large. So second price all-pay auctions can be quite unfair, which advocates for regulation.

7. Concluding remarks

The mixed Nash equilibrium in the continuous game (which is close to the mixed Nash equilibrium in the discrete game for V large and odd), has an atom on M, which seems to be a fact observed in reality. People, by bidding M are sure to win (at least half the time) the auction and they get a positive payoff each time they meet people who play less than V. Yet the continuous Nash equilibrium has no atom on 0, which is less convincing, in that bidding 0 is the best answer to an opponent who bids more than V. Given that there is an atom on M, one may expect an atom on 0. Best reply matching agrees with this fact but it doesn't assign an atom on M: the probability assigned to 0 is shared among the bids higher than V. Generalized best reply matching allows atoms on both 0 and M, but allows also other distributions.

Class room experiments show that low bids, except 0, are not often played. Students realize that when they play low bids different from 0, the opponent may bid more and make money whereas they lose their money. In some way, bidding a low amount is seen as a way to encourage the opponent to bid more, even if there is no sequentiality in this game. This way to play clearly contradicts the smooth decreasing probabilities of the mixed Nash equilibrium distribution. In some way, people play in a more bimodal way: they alternate between bid 0 (the best response of a cautious player who fears hotheads who bid a lot), and bids higher than V (to be sure to win against cautious players who do not bid a lot).

Best reply matching better fits with real behaviour. Yet, if one agrees with best reply matching reasoning –so if one no longer works with the mixed Nash equilibrium which never leads to lose money on average- , then one has to observe that second price all-pay auctions may be dangerous games.

As a matter of fact, for M not too far from V, a player can get a higher payoff with a best reply matching equilibrium than with the mixed Nash equilibrium. But, when M becomes large, more exactly when M/V becomes large, then players lose 1/12th of their budget, and even up to 1/4th of the budget with some bids. In the paper, we also draw attention to generalized best reply matching, which provides a greater degree of flexibility in choosing best responses in case of several best responses to a given strategy. It allows to combine best reply matching and focal points and better fits with the students' behaviour in the second experiment, where the students combine best reply matching and a focus on the bids 0, V and M. We show in the paper that generalized best reply matching equilibrium payoffs change with the choice of the focal values. We namely highlight that if M/V becomes larger than a given threshold, then players should delete M from their focal values, something which may be difficult given the inertia linked to focal points. This result, as well as the result about the loss of 1/12th of the budget, advocate for some regulation of the second price all-pay auctions. Yet let us make a last remark: there is a difference between the existence of a limit budget, even if this limit goes to ∞, and the non-existence of a limit budget. To establish the result in

proposition 5, we need the existence of M. And of course, for M to be a focal bid, M has to exist. Well, perhaps the best way to avoid focusing on M and to avoid a huge waste of money is to delete M, so to set no limits for the bids. That's an open question.

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Appendix 1 Proof of result 1 (out of Umbhauer 2016)

All bids between M-V/2 and M are weakly dominated by M, so it is conjectured that the Nash equilibrium strategy is a density function f(.) that decreases from 0 to M-V/2 and has an atom on M.

Call $f_2(.)$ player 2's equilibrium strategy. Suppose that player 1 plays b. She wins the auction each time player 2 bids less than b. So she gets:

$G(b) = M + \int_0^b (V - x) f_2(x) dx - b \left(\int_b^{M-V/2} f_2(x) dx + f_2(M) \right)$

We check that a player gets the same payoff with M and M-V/2, regardless of the opponents' equilibrium distribution.

$G(M) = M + \int_0^{M-V/2} (V - x) f_2(x) dx + \left(\frac{V}{2} - M \right) f_2(M) = G(M-V/2)$

G(b) has to be constant for each b in [0, M-V/2] ∪ {M}. So G'(b) = 0 for b in [0, M-V/2] .

We get $(V-b) f_2(b) - F_2(M-V/2) + F_2(b) - f_2(M) + b f_2(b) = 0$

where $F_2(.)$ is the cumulative distribution of the density function $f_2(.)$.

By construction $f_2(M) = 1 - F_2(M-V/2)$, so we get the differential equation: $V f_2(b) - 1 + F_2(b) = 0$ whose solution is: $F_2(b) = 1 + K e^{-b/V}$ where K is a constant determined as follows:

$F_2(0)=0$ because there is no atom on 0, so $1+K=0$ and $K=-1$.

It follows $F_2(b) = 1- e^{-b/V}$ for b in [0, M-V/2], $f_2(M) = 1-F_2(M-V/2) = e^{1/2-M/V}$ (<1) ,

$f_2(b) = e^{-b/V}/V$ for b in[0, M-V/2] (and $f_2(b)=0$ for b in]M-V/2,M[ )

By symmetry, we get $f_1(b) = e^{-b/V}/V$ for b in[0, M-V/2], $f_1(M) = 1-F_2(M-V/2) = e^{1/2-M/V}$ (and $f_1(b)=0$ for b in]M-V/2,M[ )

Appendix 2   Proof of the Corollary of the folk result

If there is no limit budget, player 1's payoff when she plays b is:

$G(b) = M+\int_0^b (V - x)f_2(x)dx - b(\int_b^{+\infty} f_2(x)dx$

G(b) has to be constant for each b in $[0, +\infty]$. So G'(b) = 0 for b in $[0, +\infty[$.

We get $(V-b)f_2(b) -(1-F_2(b))+bf_2(b) = 0$

i.e.: $Vf_2(b)-1 +F_2(b)=0$

So we get the solution $f_2(b) = e^{-b/V}/V$ for b in$[0, +\infty[$ By symmetry, we get $f_1(b) = e^{-b/V}/V$ for b in$[0, +\infty[$,

Appendix 3 Proof of proposition 1

- Each bid i, i from 1 to V+1 is a best reply to all bids j, j from 0 to i-1.
- Each bid i, i from V+1 to M is a best reply to all bids j, j from 0 to V.
- Bid 0 is a best reply to V and is the only best reply to bid j, j from V+1 to M

So, given that we look for a symmetric BRM equilibrium, we get:

$q_0= q_V/(M-V+1)+\sum_{i=1}^{M-V} q_{V+i}$

$q_1= q_0/M$

$q_2= q_0/M +q_1/(M-1)$

$q_i= \sum_{j=0}^{i-1} q_j/(M - j)$ i from 1 to V

$q_i= \sum_{j=0}^{i-1} q_j/(M - j)$ +q_V/(M-V+1)   i   from V+1 to M

It follows that $q_1= q_0/M$

$q_2= q_0/M +q_0/(M-1)M= q_0/(M-1)$

$q_3= q_0/M +q_1/(M-1)+q_2/(M-2)=$ $q_0/(M-1)+ q_0/(M-1)(M-2)= q_0/(M-2)$

By recurrence, if $q_i= q_0/(M-i+1)$,

$q_{i+1} = q_i +q_i/(M-i) = q_0/(M-i+1)+ q_0/[(M-i+1)(M-i)]= q_0/(M-i)$ for i from 1 to V-1

$q_i= q_0/(M-V+1)+q_V/(M-V+1)=$ $q_0/(M-V+1)+q_0/(M-V+1)^2= q_0(M-V+2)/(M-V+1)^2$ for i from V+1 to M

And $q_0+ q_0/M+ q_0/(M-1)+...q_0/(M-V+1)+ q_0-q_0/(M-V+1)^2 =1$

So $q_0(2+\sum_{i=0}^{V-1} 1/(M - i) -1/(M-V+1)^2) =1$

Appendix 4 Proof of proposition 2

For M>2V, we know that $q_0>1/3$, so q_0 is higher than the NE probability which goes to 1/V when V is large enough.

The result remains true for V≤M<2V:

$q_0= 1/(2+\sum_{i=0}^{V-1} 1/(M - i) -1/(M-V+1)^2)$ so we get $q_0= 1/(2+1/M+1/(M-1)+...+1/(M-V+1)-1/(M-V+1)^2)$ which we approximate by $1/(2+\ln(M/(M-V+1))-1/(M-V+1)^2)$

Set M= aV with a≥1. We show that $1/(2+\ln(M/(M-V+1))-1/(M-V+1)^2)>1/V$

i.e. $\ln(a/(a-1+1/V)<V-2+1/(V(a-1)+1)^2$

$\ln(a/(a-1+1/V))$ is decreasing in a. For a $=1$, $\ln(a/(a-1+1/V))= \ln(V)$, and $\ln(V)<V-2+1$ regardless of $V>1$. More generally, $\ln(a/(a-1+1/V)) <\ln(V)<V-2$ for V higher or equal to 4, so $\ln(a/(a-1+1/V)) <\ln(V) <V-2+1/(V(a-1)+1)^2$ regardless of a (≥ 1).

Appendix 5 proof of proposition 3 and 4
Proof of proposition 3
We get $\sum_{i=1}^{V-1} q_i=q_0$ $(1/(V+2))+1/(V+3)+\dots 1/2V)$ which can be approximated by $q_0\ln(2V/(V+2))$, close to $\ln(2)q_0$ if V is large .

We also get $\sum_{i=V}^{M} q_i= q_0+q_0$ $(M-V)/(M-V+1)^2=q_0(1+V/(V+1)^2)$ close to q_0 for large values of V.

So we get $q_0\simeq 1/(2+\ln(2))=0.371$, $\sum_{i=V}^{M} q_i\simeq 0,371$ and $\sum_{i=1}^{V-1} q_i=0.258$

Proof of proposition 4
For M=V, $q_0= 1/(2+1/V+1/(V-1)+1/(V-2)+\dots 1-1)\approx 1/(1+\ln(V)+\gamma)$ where γ is Euler's constant 0.577, if we approximate $1+1/2+\dots 1/V$ by $\int_1^V 1/x\,dx +\gamma = \ln(V)+\gamma$

Appendix 6 proof of proposition 5

First part of the proposition.
We calculate the mean payoff for each player when M is large. We already know that, when M is large (say M→∞) and V is a constant, then q_0 goes to ½ , $\sum_{i=1}^{V} q_i$ goes to 0, $\sum_{i=V+1}^{M} q_i$ goes to ½, which means, given that each action is played with the same probability, that $q_i=$ a=1/(2(M-V)) for each i from V+1 to M.

We look for the payoff obtained with each played message. We forget the bids from 1 to V, given that they lead to a payoff which is a constant that will be multiplied by a probability (to play the message) that is so small that even the sum of the payoffs obtained with these bids goes to 0 (because each q_i goes to 0 and $\sum_{i=1}^{V} q_i=0$.) For similar reasons we forget the payoff a player gets when she meets a player who plays a bid from 1 to V, because the sum of the payoffs is multiplied by 0. So we start by looking for the payoff obtained with bid 0, then the payoff obtained with bid V+i, i from 1 to M-V, and then we calculate the mean payoff. We add the budget at the end of the calculation.

Payoff obtained with bid 0= $q_0V/2= V/4$

Payoff obtained with bid V+1 =

$q_0V+a(V/2-V-1)-(V+1)(M-V-1)a= q_0V+a(V/2)- a(V+1)(M-V)$

Payoff obtained with bid V+2 $=q_0V-a+a(V/2-V-2)-(V+2)(M-V-2)a=$ $q_0V-a+a(V/2)-a(V+2)(M-V-1)$

Payoff obtained with bid V+i = $q_0V- a -2a-\dots-(i-1)a+a(V/2) - a(V+i) (M-V-i+1)$ i from 2 to M-V

So, for i from 1to M-V, the payoff with bid V+i is equal to:

$V/2 -a(i-1)i/2-aV(M-V+1/2)-ia(M-2V+1)+ai^2= V/2 +ai^2/2 -aV(M-V+1/2)-ia(M-2V+1/2)$.

Now we calculate the mean payoff, by multiplying V/4 by q_0, each payoff with bid V+i by a, and by summing these payoffs from i=1 to i=M-V. So we get:

$V/8 + a(M-V)V/2 - a^2(M-V)V(M-V+1/2) + a \sum_{i=1}^{M-V}(\frac{ai^2}{2} - ia\left(M - 2V + \frac{1}{2}\right))$.

$a \sum_{i=1}^{M-V}(\frac{ai^2}{2} - ia\left(M - 2V + \frac{1}{2}\right)) =$

a^2 (M-V)(M-V+1)(2M-2V+1)/12 - a^2(M-V)(M-V+1)(M-2V+1/2)/2 which goes to
(2M-2V+1)/48 - (M-2V+1/2)/8= (-4M+10V-2)/48 because a=1/(2(M-V))
a(M-V)V/2- a^2(M-V)V(M-V+1/2) goes to V/4 –V/4 because a=1/(2(M-V)).
So the mean payoff goes to:
V/8 +V/4 –V/4 + (-4M+10V-2)/48= (-2M+8V-1)/24 to which we have to add the initial budget M. This amounts to saying that, for M very large (in comparison to V), the player loses 1/12th of the budget.

Second part of the proposition,
M is now any integer higher or equal to V. So, when you bid more than V, you may get a significant payoff in front of bids lower than V+1. Yet this payoff is the same for each bid V+i, i from 1 to M-V. So we can forget it when looking for the difference of payoffs between two consecutive bids V+i and V+i+1. This difference is:
a(i+1)^2/2 -(i+1)a(M-2V+1/2)- ai^2/2 + ia(M-2V+1/2) =
(a+2ia)/2-a(M-2V+1/2)= a(i-(M-2V)). It immediately follows that for M≤2V, the payoff is growing with V+i, i from 1 to M-V, and that for M>2V, the payoff decreases with V+i, i from 1 to M-2V and then increases from i=M-2V to i=M-V.
For M large (say M→∞) and V constant, the lowest payoff, obtained with V+M-2V, i.e. M-V, is equal to: V/2 +ai^2/2 -aV(M-V+1/2)-ia(M-2V+1/2)=
V/2 +a(M-2V)^2/2 -aV(M-V+1/2)-(M-2V)a(M-2V+1/2) which goes to –M/4 when M is very large (given that a= 1/(2(M-V))).